Strong Dynamical Localization for the almost Mathieu model

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Abstract

In this note we prove Strong Dynamical Localization for the almost Mathieu operator $H_{\theta,\lambda,\omega} = -\Delta + \lambda\cos(2\pi(\theta + x\omega))$ for all $\lambda > 2$ and Diophantine frequencies ω. This improves the previous known result [22, 13] which established Dynamical Localization for a.e. θ and for $\lambda \geq 15$.

1 Introduction and main result

In this paper we prove the strong version of Dynamical Localization for the almost Mathieu operator $H_{\theta,\lambda,\omega}$, acting on $\ell^2(\mathbb{Z})$,

$$(H_{\theta,\lambda,\omega}u)(x) = u(x-1) + u(x+1) + \lambda\cos(2\pi(\theta + x\omega))u(x),\ x \in \mathbb{Z}, \tag{1}$$

with $\lambda > 2$, $\omega \in \mathbb{R}\backslash\mathbb{Q}$, and $\theta \in \mathbb{T} = [0,1[$. We shall also use the shorter notation H_θ instead of $H_{\theta,\lambda,\omega}$. More precisely our result holds for frequencies $\omega \in \mathbb{R}\backslash\mathbb{Q}$, meaning that there exist $c(\omega) > 0$ and $a(\omega) > 1$ such that

$$\forall\ j \neq 0, \quad |\sin 2\pi j\omega| > \frac{c(\omega)}{|j|(\log|j|)^{a(\omega)}}. \tag{2}$$

It is well knowm that a.e. ω satisfies (2) (see *e.g.* [23]). Throughout this paper we shall call such frequencies ω Diophantine.

Pure point spectrum with exponentially decaying eigenfunctions for a.e. θ, $\lambda > 2$ and Diophantine ω was proved in [21]. While the latter property is often referred to as Anderson

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localization, the physical understanding of localization depends on the dynamical properties, particularly the nonspreading of initially localized wave-packets under the Schrödinger time-evolution. One, generally accepted, formulation of it is, for any $q > 0$, and ψ with fast decay at ∞,

$$\sup_t \langle\langle |X|^q \rangle\rangle_{\psi,H,t} \leq C_{\psi,q}, \tag{3}$$

where the moments of the position operator X are defined by

$$\langle\langle |X|^q \rangle\rangle_{\psi,H,t} = \langle e^{-iHt}\psi,\ |X|^q e^{-iHt}\psi \rangle,$$

or if necessary using the initial state $E_H(I)\psi$ rather than ψ, where $E_H(I)$ is the spectral projector of H onto the interval I. One refers to the bound (3) as Dynamical Localization.

For ergodic families of operators H_θ, $\theta \in \Theta$ (denoting by $\mathbb{P}$ the probability measure), the constant $C_{\psi,q}$ may also depend on the parameter θ: $\sup_t \langle\langle |X|^q \rangle\rangle_{\psi,H_\theta,t} \leq C_{\psi,\theta,q}$. The following stronger form of (3) then makes sense.

Definition 1.1. *We shall say that the family $(H_\theta)_{\theta\in\Theta}$ acting on a Hilbert space $\mathcal{H}$ is strongly dynamically localized, if, for any $q > 0$ and initial state $\psi \in \mathcal{H}$, $\|\psi\| = 1$, that decays faster than any polynomial, there exists a constant $C(q,\psi) < \infty$ such that*

$$\int_\Theta \mathrm{d}\mathbb{P}(\theta) \sup_t \langle\langle |X|^q \rangle\rangle_{\psi,H_\theta,t} \leq C(q,\psi). \tag{4}$$

If necessary we can restrict ourselves to an interval of energies I, and then work with the initial state $E_{H_\theta}(I)\psi$ instead of ψ. This will define Strong Dynamical Localization on the interval I.

While the pure point spectrum follows from the Dynamical Localization (see, e.g. [5]), an example in [9, 10] shows that Anderson localization (pure point spectrum with exponentially decaying eigenfunctions) may coexist with almost ballistic dynamics (which is the worst possibility, according to [28]). Therefore, proving Dynamical Localization requires going further in the proof of pure point spectrum. The Aizenman-Molchanov technique ([1] [2] and ref. therein), wherever it works, leads naturally to a strong bound of the form (4). However, other treatments are needed for continuous or deterministic models. In [10], a property called SULE has been introduced. This property establishes a control on the size of the support of the eigenfunctions in terms of the centers of localization, $x_{E,\theta}$, namely $|\varphi_E^\theta(x)| \leq C_\varepsilon e^{\varepsilon|x_{E,\theta}|} e^{-\gamma_0|x-x_{E,\theta}|}$, for any $\varepsilon > 0$ and it is shown to entail Dynamical Localization [10]. In [14] SULE has been derived for a large class of random Schrödinger operators to which the multi-scale analysis [12] of the form [11] applies. Thereby, Dynamical Localization was proved for those models. Very recently, in [6], this result has been strenghened to *Strong* Dynamical Localization, namely it is shown that for random Hamiltonians the property SULE in the form obtained in [14] actually implies (4). In [15], an even more straightforward argument is given. We mention that these results also work in case of the random dimer operator [8], a discrete 1-D model where the Aizenman-Molchanov approach does not apply.

Besides random, another important class of ergodic operators for which results on the pure point spectrum have been obtained is the class of quasi-periodic operators. As well as with the point spectrum itself, establishing Dynamical Localization in this case requires approaches that are quite different from that of the random case.

From the dynamical point of view, for quasi-periodic models, the motion is known to be quasi-ballistic for a generic set of frequencies ω [7] (it was proved for Liouville frequencies and for the almost Mathieu model in [25]). Dynamical Localization (the *a.e.* θ version) for the almost Mathieu operator with $\lambda \geq 15$, Diophantine frequencies, has been obtained in [22] by establishing SULE. A somewhat weaker version of Dynamical Localization was independently obtained in [13] by establishing some weaker property on the decay of the eigenfunctions. Both these proofs were, in a certain sense, "second iterations" of the proof in [19], and neither extends automatically to the general $\lambda > 2$ case, nor to the strong form (4). In this paper we achieve both these goals for the almost Mathieu model. Our main result is the following:

Theorem 1.1. *Let $\lambda > 2$ and let ω be Diophantine. Then the family $(H_{\theta,\lambda,\omega})_{\theta\in\mathbb{T}}$ is strongly dynamically localized.*

Remarks:
1) We note that our proof can be adjusted to establish SULE. One can then adapt the approach of [6] and use our Lemma 2.1 in order to get the strong form of Dynamical Localization stated in Theorem 1.1. We, however, bypass the SULE condition and provide a more straightforward proof which is closer to the spirit of [13]. The idea is that the Strong Dynamical Localization of the family $(H_{\theta,\lambda,\omega})_{\theta\in\mathbb{T}}$ can be obtained in a very natural way using the (usually rather unexploited) properties of the BKG eigenfunctions expansion [3] [27] [24]. The latter is developed for some families of random operators like Schrödinger and Classical Waves in [15]. In the particular setting of the present model (a discrete one-dimensional case with simple pure point spectrum) the arguments become even more direct, since one can skip the explicit use of that expansion in eigenfunctions by roughly reconstructing it "by hand". See Section 2 for further comments.

2) We use a "restrictive" Diophantine condition (2) on ω so that we can quote directly a Lemma from [16], where the same condition is used. For the rest of the argument a weaker condition on ω, of the form $|\sin 2\pi j\omega| > \frac{c}{|j|^r}$ for some $c > 0, r > 1$, would suffice. We note that the proof of Lemma 4.1 in [16](which is the only statement we use) extends with obvious changes to frequencies ω as above with $r < 2$ (the statement becomes weaker but still sufficient for our purposes). Therefore, Theorem 1.1 holds for such values of ω as well.

3) Finally we make an additional remark that in the setting of quasi-periodic potential the result (4) - in comparison to (3) - has another physical interest. It has been argued by Piechon [26], for the Fibonacci chain, that after averaging the quantity $\langle\langle|X|^2\rangle\rangle_{\psi,H_\theta,t}$ over the parameter θ, one could get some ballistic motion. The fairly heuristic argument relies on the idea that averaging over θ of the spectral measure μ_θ of H_θ one would, somehow, recover the density of states, which is continuous (at least for the almost Mathieu model with Diophantine frequency: see [17] together with [21]). Then by Guarneri's arguments [18] [25] $\int_{\mathbb{T}} \mathrm{d}\theta \left(\frac{1}{T}\int_0^T \mathrm{d}t\langle\langle|X|^2\rangle\rangle_{\psi,H_\theta,t}\right)$ would behave like T^2. Theorem 1.1 then tells that this reasoning cannot hold for the almost Mathieu model with Diophantine conditions.

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2 Proof of Strong Dynamical Localization

We shall denote by $H_{[x,y],\theta}$ the restriction of H_θ to the interval $[x,y]$ with zero boundary conditions at $x-1$ and $y+1$, and by $G^\theta_{[x,y]}(E,\cdot,\cdot)$ the corresponding Green function, that is the inverse of $H_{[x,y],\theta}-E$. We need the following definition.

Definition 2.1. *Let $\gamma>0$, $k>0$, and $E\notin\sigma(H_{[x,y],\theta})$. A point $z\in\mathbb{Z}$ is called (γ,E,θ,k)-regular at energy E if there exists an interval $[x,y]$, with $|x-y|\le k$, containing z and such that*

$$\left|G^\theta_{[x,y]}(E,z,u)\right| < e^{-\gamma k},\ u=x,y.$$

We also recall the following well known identity. Let $z\in[x,y]$, $E\notin\sigma(H_{[x,y],\theta})$, and $\varphi\in\ell^2(\mathbb{Z})$ such that $H_\theta\varphi=E\varphi$. One then has

$$\varphi(z) = -G^\theta_{[x,y]}(E,z,x)\varphi(x-1) - G^\theta_{[x,y]}(E,z,y)\varphi(y+1). \tag{5}$$

The proof of Theorem 1.1 relies on the following lemma:

Lemma 2.1. *Assume that the hypotheses of Theorem 1.1 hold. Pick $\alpha\in(1,2)$ and any $s>\alpha$. Define $\gamma_0=\log\frac{\lambda}{2}>0$. Then there exist $L^*(\alpha,s,\omega)$, and for all $L>L^*(\alpha,s,\omega)$ and $y\in\mathbb{Z}$ a set $\Theta_L(y)\subset T$ of Lebesgue measure less than $8L^{-s+\alpha}$ such that, for any $\theta\notin\Theta_L(y)$, the following holds for all $E\in\sigma(H_\theta)$:*

$$\left(\frac{L+1}{2}\le|x-y|<L^\alpha\right)\Longrightarrow(\text{either } x \text{ or } y \text{ is } (\gamma_0/2,E,\theta,L)-\text{regular}). \tag{6}$$

Lemma 2.1 above is a modified and improved version of Lemma 4 in [21]; it is also stated in a form that is closer to the core result of the multiscale analysis given in [11]. In section 3 we shall state another version of Lemma 2.1: Lemma 3.1, which is closer to the one that can be found in [21]. We then show how to exploit the existing results to get Lemma 2.1.

The main difference between Lemma 2.1 and Lemma 4 in [21] (or equivalent lemmas in [19, 20]) is that Lemma 4 in [21] supplies a scale L^* that depends on θ *and* E. It is quite clear that one should be able to get rid of this double dependency in order to get the announced result, which was not possible in [21]. This is achieved in Section 3 below which is devoted to the proof of Lemma 2.1. The idea of Lemma 2.1 is to avoid the Borel-Cantelli Lemma systematically used in the previous proofs of Localization [19, 20, 21, 22, 13], and to work with finite scales L, supplying a set $\mathbb{T}\backslash\Theta_L(y)$ with Lebesgue measure close to one but not one, on which the conclusion (6) holds. The point is that this construction can be done for L large enough, large enough not depending on θ (nor E).

We finally point out that the Lebesgue measure of the set $\Theta_L(y)$, which consists of a y-independent part and a shift by $-y\omega$ of another y-independent part, (see (20)-(21) below), does not depend on y. This is crucial for our proof of Strong Dynamical Localization (see e.g. Eq. (15)) .

We turn to the proof of Strong Dynamical Localization: one has to handle sums of the form $\sum_{E^\theta}|\varphi^\theta_E(x)\varphi^\theta_E(y)|$. The basic idea is that for suitable x,y (6) of Lemma 2.1 asserts that either $\varphi^\theta_E(x)$ or $\varphi^\theta_E(y)$ is exponentially small. The other one can then be bounded by 1 (since $\varphi^\theta_E\in\ell^2(\mathbb{Z})$). However, one then ends up with an infinite sum over the energies E^θ, which diverges. To overcome the issue of an infinite summation, it has been proposed in [6], in the setting of random operators, to use the more detailed information concerning the exponential decay of the eigenfunctions proved in [14] (namely the property SULE mentioned above).

Below we use a quite different and more straightforward method that relies on ideas developed in [13]. We show that a very natural way to get a bound for $\sum_{E^\theta} |\varphi_E^\theta(x)\varphi_E^\theta(y)|$ is to replace φ_E^θ by the generalized eigenfunctions $\widetilde{\varphi}_E^\theta$ that supplies the eigenfunctions expansion *à la* Berezanskii [3]. This has two crucial consequences. The first one is that the generalized eigenfunctions $\widetilde{\varphi}_E^\theta$ are polynomially bounded, and this, uniformly in energy ($|\widetilde{\varphi}_E^\theta(x)| \leq (1+|x|)^\delta$, see (8) below). The exponential decay of products of the form $|\widetilde{\varphi}_E^\theta(x)\widetilde{\varphi}_E^\theta(y)|$ provided by Lemma 2.1 will not then be affected by that change. Second, and crucial, the sum over E^θ now converges. In other words, the loss of control on the eigenfunctions ($|\widetilde{\varphi}_E^\theta(x)| \leq (1+|x|)^\delta$ instead of $|\varphi_E^\theta(x)| \leq 1$) leads to a gain in the sum over E^θ. These three points (the definition of φ_E^θ and its two main consequences) are made more precise in respectively (7), (8) and (9) below.

We also note that although the spirit of this construction comes from the BKG expansion, there is no need to develop this theory in its full generality in the present setting. Indeed, since in our case the spectrum of H_θ is known to be pure point and simple, one can define directly those polynomially bounded eigenfunctions $\widetilde{\varphi}_E^\theta$ and work with them. The reader will find the general argument in [15] together with its application to classical waves and random Schrödinger operators. We finally point out that the general argument given in [15] allows one to get Strong Dynamical Localization directly, without proving Anderson Localization first.

We turn now to the proof and start with gathering the necessary ingredients. From [21] we know that under the hypotheses of the theorem the spectrum of $H_{\omega,\lambda,\theta}$ is pure point for all θ in a set $\widetilde{\Theta}$ of full Lebesgue measure [21]. In addition the spectrum is simple by a usual 1-D Wronskian argument. Let us denote by φ_E^θ the orthonormalized eigenfunctions of $H_{\omega,\lambda,\theta}$, with the energy E^θ, $\theta \in \widetilde{\Theta}$. Furthermore, as in [13], define

$$\widetilde{\varphi}_E^\theta := \varphi_E^\theta / \|B\varphi_E^\theta\|_{\ell^2}, \quad \text{so that} \quad \varphi_E^\theta = \|B\varphi_E^\theta\|_{\ell^2}\widetilde{\varphi}_E^\theta, \tag{7}$$

where B is the multiplication operator by $b(x) := (1+|x|)^{-\delta}$, $\delta > 1/2$. Note that $\widetilde{\varphi}_E^\theta$ is still in $\ell^2(\mathbb{Z})$. As briefly mentioned above, what makes those eigenfunctions of particular interest is the following: first, $\|B\widetilde{\varphi}_E^\theta\|_{\ell^2} = 1$ so that uniformly in θ and in energy E^θ, the following holds :

$$\forall\, x \in \mathbb{Z},\ |\widetilde{\varphi}_E^\theta(x)| \leq (1+|x|)^\delta. \tag{8}$$

Second, while considering a sum of the form $\sum_{n\geq 0} |\varphi_E^\theta(x)\varphi_E^\theta(y)|$ a factor $\|B\varphi_E^\theta\|_{\ell^2}^2$ will come out, and one notes that for all $\theta \in \widetilde{\Theta}$,

$$\sum_{E^\theta} \|B\varphi_E^\theta\|_{\ell^2}^2 = \sum_{x\in\mathbb{Z}} b(x)^2 \sum_{E^\theta} |\varphi_E^\theta(x)|^2 = \|b\|_{\ell^2}^2 < \infty. \tag{9}$$

The last quantity is actually nothing but the mass $\mu(\mathbb{R})$ of the spectral measure $\mu(\Delta) := \mathrm{tr}(BP(\Delta)B)$ that appears in the BKG expansion formula [3, 27, 24].

The following Lemma shows how Lemma 2.1 together with the construction (7)-(9) provide the needed exponential decay of the quantity $|\langle \delta_x, e^{-iH_\theta t}\delta_y\rangle|$, with x and y sufficiently far from each other (but not too far!). In other words, Lemma 2.2 below shows how exclusion of singular boxes together with (7)-(9) supplies the key dynamical step that implies (Strong) Dynamical Localization.

Lemma 2.2. *Let us pick $L > L^*(\alpha, s)$ and $y \in \mathbb{Z}$. Then there exists a constant $C_1(\delta, \alpha, \gamma_0) < \infty$, such that for any $\theta \notin \Theta_L(y)$ (given by Lemma 2.1), any $E = E^\theta$, and for all $x \in \mathbb{Z}$ such that $\frac{L+1}{2} \leq |x - y| < L^\alpha$, the following holds:*

$$|\varphi_E^\theta(x)\varphi_E^\theta(y)| \leq C_1(\delta, \alpha, \gamma_0)\|B\varphi_E^\theta\|_{\ell^2}^2 (1 + |y|)^{2\delta} e^{-\gamma_0 L/4}. \tag{10}$$

As a consequence, for some constant $C_2(\delta, \alpha, \gamma_0) < \infty$, and for all $\theta \notin \Theta_L(y)$,

$$\sup_t |\langle \delta_x, e^{-iH_\theta t}\delta_y\rangle| \leq C_2(\delta, \alpha, \gamma_0)(1 + |y|)^{2\delta} e^{-\gamma_0 L/4}. \tag{11}$$

Proof of Lemma 2.2.
Let us pick x, y as in the Lemma. We first replace the normalized eigenfunctions φ_E^θ by the generalized eigenfunctions $\widetilde{\varphi}_E^\theta$:

$$|\varphi_E^\theta(x)\varphi_E^\theta(y)| = \|B\varphi_E^\theta\|_{\ell^2}^2 \; |\widetilde{\varphi}_E^\theta(x)\widetilde{\varphi}_E^\theta(y)|. \tag{12}$$

Now, from Lemma 2.1, if $\theta \notin \Theta_L(y)$, then at least one of x and y, with $\frac{L+1}{2} \leq |x - y| < L^\alpha$, is $(\gamma_0/2, E, \theta, L)$-regular. Let us denote by $u = x$ or y, the regular point, and v the other one. Then, applying the identity (5) to $\widetilde{\varphi}_E^\theta$ at the point u, and taking into account the uniform polynomial bound (8), one has

$$|\widetilde{\varphi}_E^\theta(u)| \leq 2(2 + |u| + L)^\delta e^{-\gamma_0 L/2} \leq C(1 + |u|)^\delta (1 + L)^\delta e^{-\gamma_0 L/2}. \tag{13}$$

Then we use (8) to bound the second term $\widetilde{\varphi}_E^\theta(v)$, and this leads to

$$|\widetilde{\varphi}_E^\theta(u)\widetilde{\varphi}_E^\theta(v)| \leq C(1 + |u|)^\delta (1 + |v|)^\delta (1 + L)^\delta e^{-\gamma_0 L/2}. \tag{14}$$

So, in any case, if x or y is $(\gamma_0/2, E, \theta, L)$-regular, one gets that for all $\theta \notin \Theta_L(y)$

$$|\widetilde{\varphi}_E^\theta(x)\widetilde{\varphi}_E^\theta(y)| \leq C(1 + |y|)^{2\delta}(1 + L)^{(\alpha+1)\delta} e^{-\gamma_0 L/2} \leq C_1(\delta, \alpha, \gamma_0)(1 + |y|)^{2\delta} e^{-\gamma_0 L/4}.$$

Together with (12), this proves (10). Combining (10) with (9), we obtain

$$\begin{aligned}
\sup_t |\langle e^{-iH_\theta t}\delta_x, \delta_y\rangle| &\leq \sum_{E^\theta} |\varphi_E^\theta(x)\varphi_E^\theta(y)| \leq C_1(\delta, \alpha, \gamma_0)(1 + |y|)^{2\delta} e^{-\gamma_0 L/4} \sum_{E^\theta} \|B\varphi_E^\theta\|_{\ell^2}^2 \\
&\leq C_2(\delta, \alpha, \gamma_0)(1 + |y|)^{2\delta} e^{-\gamma_0 L/4}.
\end{aligned}$$

□

Proof of Theorem 1.1.

Pick some $\alpha \in (1, 2)$, and for any given $q > 0$, take $s > \alpha(q+1)$. Lemma 2.1 then supplies a sequence of scales, $L_1 = L^*(\alpha, s)$, $L_{k+1} = (2L_k - 1)^\alpha$ (so that if $L_k = \frac{L+1}{2}$ then $L_{k+1} = L^\alpha$), and for any $y \in \mathbb{Z}$, sets $\Theta_{L_k}(y)$ such that (6) and consequently (10) hold between L_k and L_{k+1},

$k \geq 1$. Splitting the integral $\int_{\mathbb{T}} \mathrm{d}\theta \, \sup_{t\in\mathbb{R}} |\langle e^{-iH_\theta t}\delta_x, \delta_y\rangle|$ in two pieces : one over $\Theta^c_{L_k}(y)$ and the second over $\Theta_{L_k}(y)$, one first gets for all $k \geq 1$ and x, y such that $L_k \leq |x-y| < L_{k+1}$:

$$\begin{aligned}\int_{\mathbb{T}} \mathrm{d}\theta \sup_{t\in\mathbb{R}} |\langle e^{-iH_\theta t}\delta_x, \delta_y\rangle| &\leq \int_{\Theta^c_{L_k}(y)} \sup_{t\in\mathbb{R}} |\langle e^{-iH_\theta t}\delta_x, \delta_y\rangle| \mathrm{d}\theta + |\Theta_{L_k}(y)| \\ &\leq 2\pi C_2(\delta, \alpha, \gamma_0)(1+|y|)^{2\delta} e^{-\frac{\gamma_0}{4}L_k} + 8L_k^{-s+\alpha}, \end{aligned} \tag{15}$$

where we used $|\langle e^{-iH_\theta t}\delta_x, \delta_y\rangle| \leq 1$, Lemma 2.1 and Lemma 2.2. Note then that $L_k \leq |x-y| < L_{k+1} = (2L_k - 1)^\alpha$ implies that $L_k \geq \frac{1}{2}|x-y|^{\frac{1}{\alpha}}$, so that (15) yields

$$\int_{\mathbb{T}} \mathrm{d}\theta \sup_{t\in\mathbb{R}} |\langle e^{-iH_\theta t}\delta_x, \delta_y\rangle| \leq 2\pi C_2(\delta, \alpha, \gamma_0)(1+|y|)^{2\delta} e^{-\frac{\gamma_0}{8}|x-y|^{\frac{1}{\alpha}}} + 8\left(\frac{1}{2}|x-y|^{\frac{1}{\alpha}}\right)^{-s+\alpha} \tag{16}$$

Equivalently, (16) means that for some constant $C_3(\delta, \alpha, \gamma_0, s)$ depending on $L^*(\alpha, s)$, one has for all $x, y \in \mathbb{Z}$:

$$\int_{\mathbb{T}} \mathrm{d}\theta \sup_{t\in\mathbb{R}} |\langle e^{-iH_\theta t}\delta_x, \delta_y\rangle| \leq C_3(\delta, \alpha, \gamma_0, s)\left[(1+|y|)^{2\delta} e^{-\frac{\gamma_0}{8}|x-y|^{\frac{1}{\alpha}}} + |x-y|^{-\frac{s}{\alpha}+1}\right]. \tag{17}$$

One can therefore end the proof of the theorem as follows. Pick $\psi \in \ell^2(\mathbb{Z})$, $\|\psi\| = 1$, that decays faster than any polynomial. Using $|\langle e^{-iH_\theta t}\psi, \delta_x\rangle| \leq \|\psi\| = 1$, one has,

$$\begin{aligned}\int_{\mathbb{T}} \mathrm{d}\theta \sup_{t\in\mathbb{R}} \left\| |X|^{q/2} e^{-iH_\theta t}\psi \right\|^2 &\leq \int_{\mathbb{T}} \mathrm{d}\theta \sum_{x\in\mathbb{Z}} |x|^q \sup_{t\in\mathbb{R}} \left|\langle e^{-iH_\theta t}\psi, \delta_x\rangle\right|^2 \\ &\leq \sum_{x\in\mathbb{Z}} |x|^q \int_{\mathbb{T}} \mathrm{d}\theta \sup_{t\in\mathbb{R}} \left|\langle e^{-iH_\theta t}\psi, \delta_x\rangle\right| \\ &\leq \sum_{y\in\mathbb{Z}} |\psi(y)| \sum_{x\in\mathbb{Z}} |x|^q \int_{\mathbb{T}} \mathrm{d}\theta \sup_{t\in\mathbb{R}} |\langle e^{-iH_\theta t}\delta_y, \delta_x\rangle|.\end{aligned}$$

Then, by (17),

$$\begin{aligned}&\int_{\mathbb{T}} \mathrm{d}\theta \sup_{t\in\mathbb{R}} \left\| |X|^{q/2} e^{-iH_\theta t}\psi \right\|^2 \\ &\leq C_3(\delta, \alpha, \gamma_0, s) \sum_{y\in\mathbb{Z}} |\psi(y)| \sum_{x\in\mathbb{Z}} |x|^q \left[(1+|y|)^{2\delta} e^{-\frac{\gamma_0}{8}|x-y|^{\frac{1}{\alpha}}} + |x-y|^{-\frac{s}{\alpha}+1}\right].\end{aligned}$$

Since s is taken larger than $\alpha(q+1)$, the sum over x converges, and since ψ decays faster than any polynomial, so does the sum over y. □

3 Proof of Lemma 2.1

We first need some further definitions in order to state the two lemmas that will imply Lemma 2.1. We denote by $B(\theta, E)$ the one-step transfer matrix of the eigenvalue equation $H_{\theta,\lambda,\omega} u = Eu$:

$$B(\theta, E) = \begin{pmatrix} E - \lambda\cos 2\pi\theta & -1 \\ 1 & 0 \end{pmatrix}.$$

Then the k-step transfer matrix is given by

$$M_k(\theta, E) = B(\theta + (k-1)\omega, E) \cdots B(\theta + \omega, E) B(\theta, E).$$

The Lyapunov exponent $\gamma(E)$ is defined as

$$\gamma(E) := \lim_{k\to\infty} \frac{1}{k} \int_{\mathbb{T}} \log \|M_k(\theta, E)\| \mathrm{d}\theta = \inf_k \frac{1}{k} \int_{\mathbb{T}} \log \|M_k(\theta, E)\| \mathrm{d}\theta. \tag{18}$$

By the subadditive ergodic Theorem (*e.g.* [5]) the limits exist and the second equality holds. Pick any $s > \alpha$ and let us define, for $L \geq 1$

$$\widetilde{\Theta}_L := \left\{ \theta \in \mathbb{T},\ \exists |j| \leq 2L^\alpha,\quad \left| \sin 2\pi \left(\theta + \frac{j}{2}\omega \right) \right| \leq \frac{1}{L^s} \right\}. \tag{19}$$

Since for any fixed $j \in \mathbb{Z}$, $\left| \{ \theta \in \mathbb{T},\ \left| \sin 2\pi \left(\theta + \frac{j}{2}\omega \right) \right| \leq L^{-s} \} \right| \leq 2L^{-s}$, one immediately gets that $\left| \widetilde{\Theta}_L \right| \leq 8L^{-s+\alpha}$. Then define the set of "bad" angles θ associated to $L \geq 1$ and $y \in \mathbb{Z}$ as follows:

$$\Theta_L(y) = \widetilde{\Theta}_L - y\omega. \tag{20}$$

Obviously, one has

$$|\Theta_L(y)| = \left| \widetilde{\Theta}_L \right| \leq \frac{8}{L^{s-\alpha}}. \tag{21}$$

Lemma 2.1 will be a consequence of the following two lemmas.

Lemma 3.1. **[21]** *Let $\varepsilon \in (0, \gamma_0/2)$, $\alpha \in (1,2)$, $s > \alpha$, and ω be a Diophantine irrational. Let L be a positive integer and assume that for all $k \geq L$, $E \in [-2-\lambda, 2+\lambda]$ and $\theta \in \mathbb{T}$, $\|M_k(\theta, E)\| \leq e^{k(\gamma(E)+\varepsilon)}$.*
Then there exists $L_1(\varepsilon, \alpha, s, \omega)$ such that for any $L \geq L_1(\varepsilon, \alpha, s, \omega)$ and any given y and $\theta \notin \Theta_L(y)$, where $\Theta_L(y)$ is given by (20), the following holds for all $E \in [-2-\lambda, 2+\lambda]$:

$$\left(x \text{ and } y \text{ are } (\gamma(E) - \varepsilon, E, \theta, L) - \text{singular and } |x - y| > \frac{L+1}{2} \right) \Longrightarrow |x-y| > L^\alpha. \tag{22}$$

Lemma 3.2. **[4, 16]** *Let I be a compact interval and assume that ω satifies the Diophantine condition (2). Then for $\varepsilon > 0$ there exists $L(\varepsilon, \omega, I)$ such that for any $k \geq L(\varepsilon, \omega, I)$, $E \in I$ and $\theta \in \mathbb{T}$, one has*

$$\frac{1}{k} \log \|M_k(\theta, E)\| < \gamma(E) + \varepsilon\,. \tag{23}$$

We shall apply Lemma 3.2 to the interval $I = [-2-\lambda, 2+\lambda]$ which contains the spectrum of H_θ, for all $\theta \in \mathbb{T}$.

Lemma 3.1 is a sort of intermediate statement between Lemma 2.1 above and Lemma 4 of [21], and does not require any new arguments, only a precise re-reading together with some slight modifications. For this reason we state Lemma 3.1 without a proof. The difference between our Lemma 3.1 and Lemma 4 of [21] is slim but crucial. To get a control on the dependency on θ, the idea is to avoid the Borel-Cantelli Lemma that leads to a set Θ of full measure in [19, 20, 21, 22, 13], and to find for a given scale L a suitable non-resonant set of

θ with a measure close to one. This will precisely be the set $\Theta_L(y)$ (see (20) and (21)). More technically we define $\Theta_L(y)$ in such a way that the conclusion of Lemma 13 in [21] holds for any scale large enough, uniformly both in x such that $\frac{L+1}{2} \leq |x-y| < L^\alpha$ and in $\theta \notin \Theta_L(y)$. As one can see from its definition (20), the set of resonances $\Theta_L(y)$ where Lemma 13 of [21] fails does depend on y. The crucial point for our proof of Strong Dynamical Localization (see e.g. Eq. (15)) is that its *measure* does not depend on y.

Lemma 3.2 is a direct corollary of Lemma 2.1 in [4] and Lemma 4.2 in [16], at least for Diophantine frequencies satisfying (2). The aim of that lemma is to take care of the (non) dependency on the energy E of the constant L^* in Lemma 2.1.

We finally show how to derive Lemma 2.1 from these two lemmas.

Proof of Lemma 2.1.
Let α, ω, s and $\varepsilon > 0$ be as in Lemma 3.1 and $I = [-2-\lambda, 2+\lambda]$. Lemma 3.2 provides the control on $\|M_k(\theta, E)\|$ that Lemma 3.1 requires. Indeed Lemma 3.2 asserts that for any $E \in I$, $\theta \in \mathbb{T}$ and $k \geq L(\varepsilon, \omega, I)$, one has

$$\|M_k(\theta, E)\| \leq e^{k(\gamma(E)+\varepsilon)}. \tag{24}$$

This means that Lemma 3.1 applies for $L \geq L(\varepsilon, \omega, I)$. Define $\Theta_L(y)$ as in (20). Let us take ε small enough so that $\gamma_0/2 \leq \gamma_0 - \varepsilon$. Then, since for all E one has $\gamma(E) \geq \log \frac{\lambda}{2} = \gamma_0$ [5], (22) of Lemma 3.1 holds with the set of phases $\Theta_L(y)$, and for L larger than some $L^*(\alpha, s, \omega)$, but with the rate $\gamma_0/2$ instead of $\gamma(E) - \varepsilon$. This in turn implies Lemma 2.1. □

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