LOCALIZATION AT LOW ENERGIES FOR ATTRACTIVE POISSON RANDOM SCHRÖDINGER OPERATORS

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Dedicated to Stanislas Molchanov on the occasion of his 65^{th} *birthday.*

Abstract. We prove exponential and dynamical localization at low energies for the Schrödinger operator with an attractive Poisson random potential in any dimension. We also conclude that the eigenvalues in that spectral region of localization have finite multiplicity.

1. Introduction and main results

The motion of an electron moving in an amorphous medium where identical impurities have been randomly scattered, each impurity creating a local attractive potential, is described by a Schrödinger equation with Hamiltonian

$$H_X := -\Delta + V_X \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d), \tag{1.1}$$

where the potential is given by

$$V_X(x) := -\sum_{\zeta \in X} u(x-\zeta), \tag{1.2}$$

with X being the location of the impurities and $-u(x-\zeta) \leq 0$ the attractive potential created by the impurity placed at ζ. Since the impurities are randomly distributed, it is natural to model the configurations of the impurities by a Poisson process on $\mathbb{R}^d$ [LiGP, PF].

The *attractive Poisson Hamiltonian* is the random Schrödinger operator $H_{\mathbf{X}}$ in (1.1) with $\mathbf{X}$ a Poisson process on $\mathbb{R}^d$ with density $\varrho > 0$; $V_{\mathbf{X}}$ being then an *attractive Poisson random potential.* The attractive Poisson Hamiltonian $H_{\mathbf{X}}$ is an $\mathbb{R}^d$-ergodic family of random self-adjoint operators; it follows from standard results (cf. [KiM, PF]) that there exists fixed subsets of $\mathbb{R}$ so that the spectrum of $H_{\mathbf{X}}$, as well as the pure point, absolutely continuous, and singular continuous components, are equal to these fixed sets with probability one.

Poisson Hamiltonians have been known to have Lifshitz tails, a strong indication of localization, for quite a long time [DV, CL, PF, Klo2, Sz, KloP, St]. In particular, the existence of Lifshitz tails for attractive Poisson Hamiltonians is proved in [KloP]. But up to recently localization was known only in one dimension [Sto]; the multi-dimensional case remaining an open question (cf. [LMW]).

We have recently proven localization at the bottom of the spectrum for Schrödinger operators with positive Poisson random potentials in arbitrary dimension [GHK1, GHK2]. We obtained both exponential (or Anderson) localization and dynamical

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localization, as well as finite multiplicity of eigenvalues. In this article we extend these results to attractive Poisson Hamiltonians, proving localization at low energies.

Localization has been known for Anderson-type Hamiltonians [HoM, CoH, Klo1, KiSS, Klo3, GK3, AENSS]. In random amorphous media, localization was known for some Gaussian random potentials [FLM, U, LMW]. In all these case there is an "a priori" Wegner estimate in all scales (e.g., [HoM, CoH, Klo1, CoHM, Ki, FLM, CoHN, CoHKN, CoHK]).

Bourgain and Kenig's proved localization for the Bernoulli-Anderson Hamiltonian, an Anderson-type Hamiltonian where the coefficients of the single-site potentials are Bernoulli random variables [BK]. They established a Wegner estimate by a multiscale analysis using "free sites" and a new quantitative version of unique continuation which gives a lower bound on eigenfunctions. Since they obtained weak probability estimates and had discrete random variables, they also introduced a new method to prove Anderson localization from estimates on the finite-volume resolvents given by a single-energy multiscale analysis. The new method does not use the perturbation of singular spectra method nor Kotani's trick as in [CoH, SW], which requires random variables with bounded densities. It is also not an energy-interval multiscale analysis as in [DrK, FrMSS, Kl], which requires better probability estimates.

To prove localization for Poisson Hamiltonians [GHK2], we exploited the probabilistic properties of Poisson point processes to use the new ideas introduced by Bourgain and Kenig [B, BK]

Here we study attractive single-site potentials, which we write as $-u$, where u is a nonnegative, nonzero L^∞-function on $\mathbb{R}^d$ with compact support, with

$$u_- \chi_{\Lambda_{\delta_-}(0)} \le u \le u_+ \chi_{\Lambda_{\delta_+}(0)} \quad \text{for some constants } u_\pm, \delta_\pm \in]0,\infty[. \tag{1.3}$$

($\Lambda_L(x)$ denotes the box of side L centered at $x \in \mathbb{R}^d$.) It follows that H_X is essentially self-adjoint on $C_c^\infty(\mathbb{R}^d)$ and $\sigma(H_{\mathbf{X}}) = \mathbb{R}$ with probability one [CL, PF]. We show that the conclusions of [GHK2] hold in an interval of negative energies of the form $]-\infty, E_0(\varrho)]$ for some $E_0(\varrho) < 0$. We obtain both exponential (or Anderson) localization and dynamical localization, as well as finite multiplicty of eigenvalues.

For a given set B, we let χ_B be its characteristic function, $\mathcal{P}_0(B)$ the collection of its countable subsets, and $\#B$ its cardinality. Given $X \in \mathcal{P}_0(A)$ and $A \subset B$, we set $X_A := X \cap A$ and $N_X(A) := \#X_A$. We write $|A|$ for the Lebesgue measure of a Borel set $A \subset \mathbb{R}^d$. We let $\Lambda_L(x) := x + \left(-\frac{L}{2}, \frac{L}{2}\right)^d$ be the box of side L centered at $x \in \mathbb{R}^d$. By Λ we will always denote some box $\Lambda_L(x)$, with Λ_L denoting a box of side L. We set $\chi_x := \chi_{\Lambda_1}(x)$, the characteristic function of the box of side 1 centered at $x \in \mathbb{R}^d$. We write $\langle x \rangle := \sqrt{1+|x|^2}$, $T(x) := \langle x \rangle^\nu$ for some fixed $\nu > \frac{d}{2}$. By $C_{a,b,\ldots}$, $c_{a,b,\ldots}$, $K_{a,b,\ldots}$, etc., will always denote some finite constant depending only on $a, b, \ldots$.

A Poisson process on a Borel set $B \subset \mathbb{R}^d$ with density $\varrho > 0$ is a map $\mathbf{X}$ from a probability space $(\Omega, \mathbb{P})$ to $\mathcal{P}_0(B)$, such that for each Borel set $A \subset B$ with $|A| < \infty$ the random variable $N_{\mathbf{X}}(A)$ has Poisson distribution with mean $\varrho|A|$, i.e.,

$$\mathbb{P}\{N_{\mathbf{X}}(A) = k\} = \tfrac{(\varrho|A|)^k}{k!} \mathrm{e}^{-\varrho|A|} \quad \text{for } k = 0, 1, 2, \ldots, \tag{1.4}$$

and the random variables $\{N_{\mathbf{X}}(A_j)\}_{j=1}^n$ are independent for disjoint Borel subsets $\{A_j\}_{j=1}^n$ (e.g., [K, Re]).

Theorem 1.1. *Let $H_{\mathbf{X}}$ be an attractive Poisson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ with density $\varrho > 0$. Then there exists an energy $E_0 = E_0(\varrho) < 0$ for which the following holds $\mathbb{P}$-a.e.: The operator $H_{\mathbf{X}}$ has pure point spectrum in $]-\infty, E_0]$ with exponentially localized eigenfunctions, and, if ϕ is an eigenfunction of $H_{\mathbf{X}}$ with eigenvalue $E \in]-\infty, E_0]$ we have, with $m_E := \frac{1}{8}\sqrt{\frac{1}{2}E_0 - E} \le m_{E_0} := \frac{1}{8}\sqrt{-\frac{1}{2}E_0}$, that*

$$\|\chi_x \phi\| \le C_{\mathbf{X},\phi}\, e^{-m_E|x|} \quad \text{for all } x \in \mathbb{R}^d. \tag{1.5}$$

Moreover, there exist $\tau > 1$ and $s \in]0,1[$ such that for all eigenfunctions ψ, ϕ (possibly equal) with the same eigenvalue $E \in]-\infty, E_0]$ we have

$$\|\chi_x \psi\|\,\|\chi_y \phi\| \le C_{\mathbf{X}} \|T^{-1}\psi\|\|T^{-1}\phi\|\, e^{\langle y\rangle^\tau} e^{-|x-y|^s} \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{1.6}$$

In particular, the eigenvalues of $H_{\mathbf{X}}$ in $]-\infty, E_0]$ have finite multiplicity, and $H_{\mathbf{X}}$ exhibits dynamical localization in $]-\infty, E_0]$, that is, for any $p > 0$ we have

$$\sup_t \|\langle x\rangle^p e^{-itH_{\mathbf{X}}} \chi_{]-\infty,E_0]}(H_{\mathbf{X}})\chi_0\|_2^2 < \infty. \tag{1.7}$$

The proof of Theorem 1.1 relies on the construction introduced in [GHK2] for Poisson Hamiltonians, based on the new multiscale analysis of Bourgain [B] and Bourgain-Kenig [BK] for Bernoulli-Anderson Hamiltonians. Exponential localization follows as in [BK]. The decay of eigenfunction correlations given in (1.6) then follows from [GK6] as in [GHK2]. Dynamical localization and finite multiplicity of eigenvalues are consequences of (1.6).

The Bourgain-Kenig multiscale analysis requires some detailed knowledge about the location of the impurities, as well as information on "free sites", and relies on conditional probabilities. To deal with these issues and also handle the measurability questions that appear for the Poisson process, in [GHK2] we performed a finite volume reduction in each scale as part of the multiscale analysis.

In this note we review the basic construction of [GHK2], and apply it to attractive Poisson Hamiltonians. But since these are unbounded from below, we need to modify the finite volume reduction and the "a priori" finite volume estimates.

2. Attractive Poisson Hamiltonians

The Poisson process $\mathbf{X}$ on $\mathbb{R}^d$ with density ϱ is constructed from a marked Poisson process as follows: Let $\mathbf{Y}$ be a Poisson process on $\mathbb{R}^d$ with density 2ϱ, and to each $\zeta \in Y$ associate a Bernoulli random variable $\boldsymbol{\varepsilon}_\zeta$, either 0 or 1 with equal probability, with $\boldsymbol{\varepsilon}_{\mathbf{Y}} = \{\boldsymbol{\varepsilon}_\zeta\}_{\zeta\in\mathbf{Y}}$ independent random variables. Then $(\mathbf{Y}, \boldsymbol{\varepsilon}_{\mathbf{Y}})$ is a Poisson process with density 2ρ on the product space $\mathbb{R}^d \times \{0,1\}$, the *marked Poisson process*; its underlying probability space will still be denoted by $(\Omega, \mathbb{P})$. (We use the notation $(Y, \varepsilon_Y) := \{(\zeta, \varepsilon_\zeta);\, \zeta \in Y\} \in \mathcal{P}_0(\mathbb{R}^d \times \{0,1\})$.) Define maps $\mathcal{X}, \mathcal{X}' \colon \mathcal{P}_0(\mathbb{R}^d \times \{0,1\}) \to \mathcal{P}_0(\mathbb{R}^d)$ by

$$\mathcal{X}(\tilde{Z}) := \{\zeta \in \mathbb{R}^d;\, (\zeta, 1) \in \tilde{Z}\}, \quad \mathcal{X}'(\tilde{Z}) := \{\zeta \in \mathbb{R}^d;\, (\zeta, 0) \in \tilde{Z}\}, \tag{2.1}$$

for all $\tilde{Z} \in \mathcal{P}_0(\mathbb{R}^d \times \{0,1\})$. Then the maps $\mathbf{X}, \mathbf{X}' \colon \Omega \to \mathcal{P}_0(\mathbb{R}^d)$, given by

$$\mathbf{X} := \mathcal{X}(\mathbf{Y}, \boldsymbol{\varepsilon}_{\mathbf{Y}}), \quad \mathbf{X}' := \mathcal{X}'(\mathbf{Y}, \boldsymbol{\varepsilon}_{\mathbf{Y}}), \tag{2.2}$$

i.e., $\mathbf{X}(\omega) = \mathcal{X}(\mathbf{Y}(\omega), \boldsymbol{\varepsilon}_{\mathbf{Y}(\omega)}(\omega))$, $\mathbf{X}'(\omega) = \mathcal{X}'(\mathbf{Y}(\omega), \boldsymbol{\varepsilon}_{\mathbf{Y}(\omega)}(\omega))$, are Poisson processes on $\mathbb{R}^d$ with density ϱ (cf. [K, Section 5.2], [Re, Example 2.4.2]), and we have

$$N_{\mathbf{X}}(A) + N_{\mathbf{X}'}(A) = N_{\mathbf{Y}}(A) \quad \text{for all Borel sets } A \subset \mathbb{R}^d. \tag{2.3}$$

If $\mathbf{X}$ is a Poisson process on $\mathbb{R}^d$ with density ϱ, then $\mathbf{X}_A$ is a Poisson process on A with density ϱ for each Borel set $A \subset \mathbb{R}^d$, with $\{\mathbf{X}_{A_j}\}_{j=1}^n$ being independent Poisson processes for disjoint Borel subsets $\{A_j\}_{j=1}^n$. Similar considerations apply to $\mathbf{X}'$ and to the marked Poisson process $(\mathbf{Y}, \boldsymbol{\varepsilon}_{\mathbf{Y}})$, with $\mathbf{X}_A, \mathbf{X}'_A, \mathbf{Y}_A, \boldsymbol{\varepsilon}_{\mathbf{Y}_A}$ satisfying (2.2).

From now on we fix a probability space $(\Omega, \mathbb{P})$ on which the Poisson processes $\mathbf{X}$ and $\mathbf{X}'$, with density ϱ, and $\mathbf{Y}$, with density 2ϱ, are defined, as well as the Bernoulli random variables $\boldsymbol{\varepsilon}_{\mathbf{Y}}$, and we have (2.2). All events will be defined with respect to this probability space. $H_{\mathbf{X}}$ (and $H_{\mathbf{Y}}$) will always denote an attractive Poisson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ with density $\varrho > 0$ (2ϱ), as in (1.1)-(1.3).

We start by showing that the attractive Poisson Hamiltonian is self-adjoint and we have trace estimates needed in the multiscale analysis.

Proposition 2.1. *The attractive Poisson Hamiltonians $H_{\mathbf{X}}$ and $H_{\mathbf{Y}}$ are essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ with probability one. In addition, we have*

$$\operatorname{tr}\{T^{-1}\mathrm{e}^{-tH_{\mathbf{X}}}T^{-1}\} < \infty \quad \textit{for all } t > 0 \;\; \mathbb{P}\textit{-a.e.}, \tag{2.4}$$

and

$$\operatorname{tr}(T^{-1}\chi_{]-\infty,E]}(H_{\mathbf{X}})T^{-1}) < \infty \quad \textit{for all } E \in \mathbb{R} \;\; \mathbb{P}\textit{-a.e.} \tag{2.5}$$

Since the potential is attractive, it may create infinitely deep wells. This is controlled by the following estimate.

Lemma 2.2. *Given a box Λ we set*

$$\mathbb{J}_{\delta_+,\Lambda} := \left(\tfrac{1}{2}\delta_+\mathbb{Z}^d\right) \cap \Lambda. \tag{2.6}$$

There exists $L^* = L^*(d, \varrho, \delta_+)$, such that for any $L \geq L^*$ we have

$$\mathbb{P}\left\{N_{\mathbf{Y}}(\Lambda_{2\delta_+}(j))) \leq \varrho \log L \quad \forall j \in \mathbb{J}_{\delta_+,\Lambda_L}\right\} \geq 1 - L^{-\frac{2}{3}\log\log L} \tag{2.7}$$

and

$$\mathbb{P}\{\|\chi_{\Lambda_L} V_{\mathbf{Y}}\|_\infty \leq u_+ \varrho \log L\} \geq 1 - L^{-\frac{2}{3}\log\log L}. \tag{2.8}$$

It follows that for $\mathbb{P}$-a.e. ω we have

$$V_{\mathbf{X}(\omega)}(x) \geq V_{\mathbf{Y}(\omega)}(x) \geq -c_\omega \log(1 + |x|) \quad \textit{for all } x \in \mathbb{R}^d, \tag{2.9}$$

where $c_\omega > 0$ (depending also on $p, \delta_+, \varrho, u_+$).

Proof. We may assume $\varrho \log L \geq 1$. Standard bounds on Poisson random variables (cf. [GHK2, Eq. (2.7)]) give

$$\mathbb{P}\{N_{\mathbf{Y}}(\Lambda_{2\delta_+}(x)) > \varrho \log L\} \leq \left(\tfrac{2\varrho(2\delta_+)^d}{\varrho \log L}\right)^{\varrho \log L} \leq L^{-\frac{3}{4}\log\log L} \tag{2.10}$$

for any $x \in \mathbb{R}^d$, if $L \geq L_1^*(d, \varrho, \delta_+)$. It follows that for $L \geq L^*(d, \varrho, \delta_+)$ we have

$$\mathbb{P}\left\{N_{\mathbf{Y}}(\Lambda_{2\delta_+}(j)) \leq \varrho \log L \quad \forall j \in \mathbb{J}_{\delta_+,\Lambda_L}\right\} \geq 1 - \left(\tfrac{2L}{\delta_+}\right)^d L^{-\frac{3}{4}\log\log L} \geq 1 - L^{-\frac{2}{3}\log\log L}. \tag{2.11}$$

Now, for any $x \in \Lambda_L$ there exists $j \in \mathbb{J}_{\delta_+,\Lambda_L}$ s.t. $\Lambda_{\delta+}(x) \subset \Lambda_{2\delta+}(j)$. Hence, we also have

$$\mathbb{P}\{N_{\mathbf{Y}}(\Lambda_{\delta_+}(x) \cap \Lambda_L) \leq \varrho \log L \quad \forall x \in \Lambda_L\} \geq 1 - L^{-\frac{2}{3}\log\log L}. \tag{2.12}$$

But if the event in (2.12) occurs, it follows from (1.3) that

$$|V_{\mathbf{Y}}(x)| \leq u_+ \varrho \log L \quad \forall x \in \Lambda_L, \tag{2.13}$$

and (2.8) follows, since

$$0 \geq V_{\mathbf{X}}(x) \geq V_{\mathbf{Y}}(x) \tag{2.14}$$

because of (2.2). The Borel-Cantelli Lemma now gives (2.9). □

In the one-dimensional case an estimate similar to (2.9) can be found in [HW].

Proof of Proposition 2.1. In view of (2.9), it follows from the Faris-Levine Theorem [RS, Theorem X.38] that $H_{\mathbf{X}}$ and $H_{\mathbf{Y}}$ are essentially self-adjoint on $\mathcal{C}_c^\infty(\mathbb{R}^d)$ with probability one.

The trace estimate (2.4) follows from Gaussian bounds on heat kernels [BrLM, Lemma 1.7], which hold $\mathbb{P}$-a.e. in view of (2.9). As a consequence, we have

$$\begin{aligned} \operatorname{tr}(T^{-1}\chi_{]-\infty,E]}(H_{\mathbf{X}})T^{-1}) &= \operatorname{tr}(T^{-1}\mathrm{e}^{-H_{\mathbf{X}}}\mathrm{e}^{2H_{\mathbf{X}}}\chi_{]-\infty,E]}(H_{\mathbf{X}})\mathrm{e}^{-H_{\mathbf{X}}}T^{-1}) \\ &\leq \mathrm{e}^{2E}\operatorname{tr}(T^{-1}\mathrm{e}^{-2H_{\mathbf{X}}}T^{-1}) < \infty \quad \text{for all } E \in \mathbb{R} \ \ \mathbb{P}\text{-a.e.} \end{aligned} \tag{2.15}$$

□

Given two disjoint configurations $X, Y \in \mathcal{P}_0(\mathbb{R}^d)$ and $t_Y = \{t_\zeta\}_{\zeta \in Y} \in [0,1]^Y$, we set

$$H_{X,(Y,t_Y)} := -\Delta + V_{X,(Y,t_Y)}, \text{ where } V_{X,(Y,t_Y)}(x) := V_X(x) + \sum_{\zeta \in Y} t_\zeta u(x-\zeta). \tag{2.16}$$

In particular, given $\varepsilon_Y \in \{0,1\}^Y$ we have, recalling (2.1), that

$$H_{X,(Y,\varepsilon_Y)} = H_{X \sqcup \mathcal{X}(Y,\varepsilon_Y)}. \tag{2.17}$$

We also write $H_{(Y,t_Y)} := H_{\emptyset,(Y,t_Y)}$ and

$$H_\omega := H_{\mathbf{X}(\omega)} = H_{(\mathbf{Y}(\omega),\boldsymbol{\varepsilon}_{\mathbf{Y}(\omega)}(\omega))}. \tag{2.18}$$

3. Finite volume

The multiscale analysis requires finite volume operators,which are defined as follows. Given a box $\Lambda = \Lambda_L(x)$ in $\mathbb{R}^d$ and a configuration X, we set

$$H_{X,\Lambda} := -\Delta_\Lambda + V_{X,\Lambda} \quad \text{on} \quad \mathrm{L}^2(\Lambda), \tag{3.1}$$

where Δ_Λ is the the Laplacian on Λ with Dirichlet boundary condition, and

$$V_{X,\Lambda} := \chi_\Lambda V_{X_\Lambda} \quad \text{with } V_{X_\Lambda} \text{ as in (1.2)}. \tag{3.2}$$

The finite volume resolvent is $R_{X,\Lambda}(z) := (H_{X,\Lambda} - z)^{-1}$.

We have $\Delta_\Lambda = \nabla_\Lambda \cdot \nabla_\Lambda$, where ∇_Λ is the gradient with Dirichlet boundary condition. We sometimes identify $\mathrm{L}^2(\Lambda)$ with $\chi_\Lambda \mathrm{L}^2(\mathbb{R}^d)$ and, when necessary, will use subscripts Λ and $\mathbb{R}^d$ to distinguish between the norms and inner products of $\mathrm{L}^2(\Lambda)$ and $\mathrm{L}^2(\mathbb{R}^d)$. Note that we always have

$$\chi_{\widehat{\Lambda}} V_{X,\Lambda} = \chi_{\widehat{\Lambda}} V_{X,\Lambda'}, \tag{3.3}$$

where

$$\widehat{\Lambda} = \widehat{\Lambda}_L(x) := \Lambda_{L-\delta_+}(x) \quad \text{with } \delta_+ \text{ as in (1.3)}, \tag{3.4}$$

which suffices for the multiscale analysis.

The multiscale analysis estimates probabilities of desired properties of finite volume resolvents at energies $E \in \mathbb{R}$. ($L^{p\pm}$ means $L^{p\pm\delta}$ for some small $\delta > 0$. We will write $\sqcup$ for disjoint unions: $C = A \sqcup B$ means $C = A \cup B$ with $A \cap B = \emptyset$.)

Definition 3.1. *Consider an energy $E \in \mathbb{R}$ and a rate of decay $m > 0$. A box Λ_L is said to be (X, E, m)-good if*

$$\|R_{X,\Lambda_L}(E)\| \le \mathrm{e}^{L^{1-}} \tag{3.5}$$

and

$$\|\chi_x R_{X,\Lambda_L}(E)\chi_y\| \le \mathrm{e}^{-m|x-y|}, \quad \text{for } x, y \in \Lambda_L \text{ with } |x-y| \ge \tfrac{L}{10}. \tag{3.6}$$

We say that the box Λ_L is (ω, E, m)-good if it is $(\mathbf{X}(\omega), E, m)$-good.

Condition (3.6) is the standard notion of a regular box in the multiscale analysis [FrS, FrMSS, DrK, GK1, Kl]. Condition (3.5) plays the role of a Wegner estimate. This control on the resolvent is just good enough so that it does not destroy the exponential decay obtained with (3.6). Since Wegner is proved scale by scale (as in [CKM, B, BK]), it is incorporated in the definition of the goodness of a given box.

But *goodness* of boxes does not suffice for the induction step in the multiscale analysis given in [B, BK], which also needs an adequate supply of *free sites* to obtain a Wegner estimate at each scale. Given two disjoint configurations $X, Y \in \mathcal{P}_0(\mathbb{R}^d)$ and $t_Y = \{t_\zeta\}_{\zeta \in Y} \in [0,1]^Y$, we recall (2.16) and define the corresponding finite volume operators $H_{X,(Y,t_Y),\Lambda}$ as in (3.1) and (3.2) using X_Λ, Y_Λ and t_{Y_Λ}, i.e.,

$$H_{X,(Y,t_Y),\Lambda} := -\Delta_\Lambda + V_{X,(Y,t_Y),\Lambda} \quad \text{with} \quad V_{X,(Y,t_Y),\Lambda} := \chi_\Lambda V_{X_\Lambda,(Y_\Lambda,t_{Y_\Lambda})}, \tag{3.7}$$

with $R_{X,(Y,t_Y),\Lambda}(z)$ being the corresponding finite volume resolvent.

Definition 3.2. *Consider two configurations $X, Y \in \mathcal{P}_0(\mathbb{R}^d)$ and an energy E. A box Λ_L is said to be (X, Y, E, m)-good if $X \cap Y = \emptyset$ and we have (3.5) and (3.6) with $R_{X,(Y,t_Y),\Lambda_L}(E)$ for all $t_Y \in [0,1]^Y$. In this case Y consists of (X, E)-free sites for the box Λ_L.* (In particular, the box Λ_L is $(X \sqcup \mathcal{X}(Y, \varepsilon_Y), E, m)$-good for all $\varepsilon_Y \in \{0,1\}^Y$.)

The multiscale analysis requires some detailed knowledge about the location of the impurities, that is, about the Poisson process configuration, as well as information on "free sites". To deal with this and also handle the measurability questions that appear for the Poisson process, a finite volume reduction was performed in [GHK2] as part of the multiscale analysis. The key is that a Poisson point can be moved a little bit without spoiling the goodness of boxes [GHK2, Lemma 3.3]. We now recall the construction of [GHK2], with some slight adaptations to our present setting.

Definition 3.3. *Let $\eta_L := \mathrm{e}^{-L^{10^6 d}}$ for $L > 0$. Given a box $\Lambda = \Lambda_L(x)$, set*

$$\mathbb{J}_\Lambda := \{j \in x + \eta_L \mathbb{Z}^d;\ \Lambda_{\eta_L}(j) \subset \Lambda\}. \tag{3.8}$$

A configuration $X \in \mathcal{P}_0(\mathbb{R}^d)$ is said to be Λ-acceptable if (recall (2.6))

$$N_X(\Lambda_{2\delta_+}(j)) < \varrho \log L \quad \forall j \in \mathbb{J}_{\delta_+,\Lambda}, \tag{3.9}$$

$$N_X(\Lambda_{\eta_L}(j)) \le 1 \quad \text{for all} \quad j \in \mathbb{J}_\Lambda, \tag{3.10}$$

and

$$N_X(\Lambda \setminus \cup_{j\in\mathbb{J}_\Lambda} \Lambda_{\eta_L(1-\eta_L)}(j)) = 0; \tag{3.11}$$

it is Λ-acceptable$'$ if it satisfies (3.9),(3.10)*, and the weaker*

$$N_X(\Lambda \setminus \cup_{j\in\mathbb{J}_\Lambda} \Lambda_{\eta_L}(j)) = 0. \tag{3.12}$$

We set

$$\mathcal{Q}_\Lambda^{(0)} := \{X \in \mathcal{P}_0(\mathbb{R}^d);\ X \text{ is } \Lambda\text{-acceptable}\}, \tag{3.13}$$

$$\mathcal{Q}_\Lambda^{(0\prime)} := \{X \in \mathcal{P}_0(\mathbb{R}^d);\ X \text{ is } \Lambda\text{-acceptable}'\}, \tag{3.14}$$

and consider the event (recall that $\mathbf{Y}$ is the Poisson process with density 2ϱ)

$$\Omega_\Lambda^{(0)} := \{\mathbf{Y} \in \mathcal{Q}_\Lambda^{(0)}\}. \tag{3.15}$$

Note that $\Omega_\Lambda^{(0)} \subset \{\mathbf{X} \in \mathcal{Q}_\Lambda^{(0)}\}$ and $\mathcal{Q}_\Lambda^{(0)} \subset \mathcal{Q}_\Lambda^{(0\prime)}$. Condition (3.11) is put in to avoid ambiguities in the multiscale analysis.

Remark 3.4. *Note that* (3.9) *is not the same as* [GHK2, Eq. (3.19)]*, and hence the above definitions of Λ-acceptable and Λ-acceptable$'$ configurations are slightly different from the ones given in* [GHK2, Definition 3.4]*. The reason for* (3.9) *is to ensure the estimates* (3.17) *and* (3.18) *in the next lemma.*

Lemma 3.5. *Let $\Lambda = \Lambda_L$. Then for all Λ-acceptable$'$ configurations Y we have*

$$N_Y(\Lambda) \le \varrho(2\delta_+)^{-d} L^d \log L, \tag{3.16}$$

$$V_Y(x) \ge -u_+\varrho \log L \quad \text{for all } x \in \Lambda, \tag{3.17}$$

and

$$H_{(Y,t_Y),\Lambda} \ge -u_+\varrho \log L \quad \text{for all } t_Y \in [0,1]^Y. \tag{3.18}$$

Proof. The estimate (3.16) is an immediate consequence of (3.9). It also follows from (3.9), by the same argument used for (2.13), that for all Λ-acceptable$'$ configurations Y we have the lower bound (3.17), from which we get (3.18) since $H_{(Y,t_Y),\Lambda} \ge H_{Y,\Lambda} \ge V_{Y,\Lambda}$. □

Lemma 3.6. *There exists a scale $\overline{L} = \overline{L}(d,\varrho,\delta_+) < \infty$, such that if $L \ge \overline{L}$ we have*

$$\mathbb{P}\{\Omega_{\Lambda_L}^{(0)}\} \ge 1 - L^{-\frac{1}{2}\log\log L}. \tag{3.19}$$

Proof. Recalling (2.7), we have

$$\mathbb{P}\{\Omega_{\Lambda_L}^{(0)}\} \ge 1 - L^{-\frac{2}{3}\log\log L} - 4d\varrho(L^{d-1} + L^d)\eta_L - 2\varrho^2 L^d \eta_L^d, \tag{3.20}$$

and hence (3.19) follows for large L. □

Lemma 3.6 tells us that inside the box Λ, outside an event of negligible probability in the multiscale analysis, we only need to consider Λ-acceptable configurations of the Poisson process Y.

Fix a box $\Lambda = \Lambda_L(x)$, then

$$X \overset{\Lambda}{\sim} Y \Longleftrightarrow N_X(\Lambda_{\eta_L}(j)) = N_Y(\Lambda_{\eta_L}(j)) \quad \text{for all} \quad j \in \mathbb{J}_\Lambda \tag{3.21}$$

introduces an equivalence relation in both $\mathcal{Q}_\Lambda^{(0\prime)}$ and $\mathcal{Q}_\Lambda^{(0)}$; the equivalence class of X in $\mathcal{Q}_\Lambda^{(0\prime)}$ will be denoted by $[X]'_\Lambda$. If $X \in \mathcal{Q}_\Lambda^{(0)}$, then $[X]_\Lambda = [X]'_\Lambda \cap \mathcal{Q}_\Lambda^{(0)}$ is its equivalence class in $\mathcal{Q}_\Lambda^{(0)}$. Note that $[X]'_\Lambda = [X_\Lambda]'_\Lambda$. We also write

$$[A]_\Lambda := \bigcup_{X \in A} [X]_\Lambda \quad \text{for subsets} \quad A \subset \mathcal{Q}_\Lambda^{(0)}. \tag{3.22}$$

The following lemma [GHK2, Lemma 3.6] tells us that "goodness" of boxes is a property of equivalence classes of acceptable′ configurations: changing configurations inside an equivalence class takes good boxes into just-as-good (jgood) boxes. Proceeding as in the lemma, we find that changing configurations inside an equivalence class takes jgood boxes into what we may call just-as-just-as-good (jjgood) boxes, and so on. Since we will only carry this procedure a bounded number of times, the bound independent of the scale, we will simply call them all jgood boxes.

Lemma 3.7 ([GHK2]). *Fix $E_0 > 0$ and consider an energy $E \in [0, E_0]$. Suppose the box $\Lambda = \Lambda_L$ (with L large) is (X, E, m)-good for some $X \in \mathcal{Q}_{\Lambda_L}^{(0\prime)}$. Then for all $Y \in [X]'_\Lambda$ the box Λ is (Y, E, m)-jgood, that is,*

$$\|R_{Y,\Lambda}(E)\| \le \mathrm{e}^{L^{1-} + \eta_L^{\frac{1}{4}}} \sim \mathrm{e}^{L^{1-}} \tag{3.23}$$

and

$$\|\chi_x R_{Y,\Lambda}(E) \chi_y\| \le \mathrm{e}^{-m|x-y|} + \eta_L^{\frac{1}{4}} \sim \mathrm{e}^{-m|x-y|}, \quad \text{for } x, y \in \Lambda \text{ with } |x-y| \ge \tfrac{L}{10}. \tag{3.24}$$

Moreover, if $X, Y, X \sqcup Y \in \mathcal{Q}_\Lambda^{(0\prime)}$ and the box Λ is (X, Y, E, m)-good, then for all $X_1 \in [X]'_\Lambda$ and $Y_1 \in [Y]'_\Lambda$ we have $X_1 \sqcup Y_1 \in [X \sqcup Y]'_\Lambda$, and the box Λ is (X_1, Y_1, E, m)-jgood as in (3.23) *and* (3.24).

s

We also have a lemma [GHK2, Lemma 3.8] about the distance to the spectrum inside equivalence classes.

Lemma 3.8 ([GHK2]). *Fix $E_0 > 0$ and consider an energy $E \in [0, E_0]$ and a box $\Lambda = \Lambda_L$ (with L large). Suppose* $\operatorname{dist}(E, \sigma(H_{X,\Lambda})) \le \tau_L$ *for some $X \in \mathcal{Q}_{\Lambda_L}^{(0\prime)}$, where $\sqrt{\eta_L} \ll \tau_L < \frac{1}{2}$. Then*

$$\operatorname{dist}(E, \sigma(H_{Y,\Lambda})) \le \mathrm{e}^{\eta_L^{\frac{1}{4}}} \tau_L \quad \textit{for all} \quad Y \in [X]'_\Lambda. \tag{3.25}$$

In view of (3.16)-(3.10) we have

$$\mathcal{Q}_\Lambda^{(0)} / \overset{\Lambda}{\sim} = \{[J]_\Lambda;\, J \in \mathcal{J}_\Lambda\}, \quad \text{where} \quad \mathcal{J}_\Lambda := \{J \subset \mathbb{J}_\Lambda;\, (3.9) \text{ holds}\}, \tag{3.26}$$

and we can write $\mathcal{Q}_\Lambda^{(0)}$ and $\Omega_\Lambda^{(0)}$ as

$$\mathcal{Q}_\Lambda^{(0)} = \bigsqcup_{J \in \mathcal{J}_\Lambda} [J]_\Lambda \quad \text{and} \quad \Omega_\Lambda^{(0)} = \bigsqcup_{J \in \mathcal{J}_\Lambda} \{\mathbf{Y} \in [J]_\Lambda\}. \tag{3.27}$$

We now introduce the basic Poisson configurations and basic events with which we will construct all the relevant probabilistic events. These basic events have to take into account in their very structure the presence of free sites and the finite volume reduction. The following definitions are borrowed from [GHK2].

Definition 3.9. *Given $\Lambda = \Lambda_L(x)$, a Λ-bconfset (basic configuration set) is a subset of $\mathcal{Q}_\Lambda^{(0)}$ of the form*

$$C_{\Lambda,B,S} := \bigsqcup_{\varepsilon_S \in \{0,1\}^S} [B \cup \mathcal{X}(S, \varepsilon_S)]_\Lambda = \bigsqcup_{S' \subset S} [B \cup S']_\Lambda, \tag{3.28}$$

where we always implicitly assume $B \sqcup S \in \mathcal{J}_\Lambda$. $C_{\Lambda,B,S}$ is a Λ-dense bconfset if S satisfies the density condition (cf. (3.4))

$$\#(S \cap \widehat{\Lambda}_{L^{1-}}) \geq L^{d-} \quad \text{for all boxes} \quad \Lambda_{L^{1-}} \subset \Lambda_L. \tag{3.29}$$

We also set

$$C_{\Lambda,B} := C_{\Lambda,B,\emptyset} = [B]_\Lambda. \tag{3.30}$$

Definition 3.10. *Given $\Lambda = \Lambda_L(x)$, a Λ-bevent (basic event) is a subset of $\Omega_\Lambda^{(0)}$ of the form*

$$\mathcal{C}_{\Lambda,B,B',S} := \{\mathbf{Y} \in [B \sqcup B' \sqcup S]_\Lambda\} \cap \{\mathbf{X} \in C_{\Lambda,B,S}\} \cap \{\mathbf{X}' \in C_{\Lambda,B',S}\}, \tag{3.31}$$

where we always implicitly assume $B \sqcup B' \sqcup S \in \mathcal{J}_\Lambda$. In other words, the Λ-bevent $\mathcal{C}_{\Lambda,B,B',S}$ consists of all $\omega \in \Omega_\Lambda^{(0)}$ satisfying

$$\begin{array}{lll} N_{\mathbf{X}(\omega)}(\Lambda_{\eta_L}(j)) = 1 & \text{if} & j \in B, \\ N_{\mathbf{X}'(\omega)}(\Lambda_{\eta_L}(j)) = 1 & \text{if} & j \in B', \\ N_{\mathbf{Y}(\omega)}(\Lambda_{\eta_L}(j)) = 1 & \text{if} & j \in S, \\ N_{\mathbf{Y}(\omega)}(\Lambda_{\eta_L}(j)) = 0 & \text{if} & j \in \mathbb{J}_\Lambda \setminus (B \sqcup B' \sqcup S). \end{array} \tag{3.32}$$

$\mathcal{C}_{\Lambda,B,B',S}$ is a Λ-dense bevent if S satisfies the density condition (3.29)*. In addition, we set*

$$\mathcal{C}_{\Lambda,B,B'} := \mathcal{C}_{\Lambda,B,B',\emptyset} = \{\mathbf{X} \in C_{\Lambda,B}\} \cap \{\mathbf{X}' \in C_{\Lambda,B'}\}. \tag{3.33}$$

The number of possible bconfsets and bevents in a given box is always finite. We always have

$$\mathcal{C}_{\Lambda,B,B',S} \subset \{\mathbf{X} \in C_{\Lambda,B,S}\} \cap \Omega_\Lambda^{(0)}, \tag{3.34}$$

$$\mathcal{C}_{\Lambda,B,B',S} \subset \mathcal{C}_{\Lambda,\emptyset,\emptyset,B \sqcup B' \sqcup S} = \{\mathbf{Y} \in [B \sqcup B' \sqcup S]_\Lambda\}. \tag{3.35}$$

Note also that it follows from (3.15), (3.26) and (3.33) that

$$\Omega_\Lambda^{(0)} = \bigsqcup_{\{(B,B');\, B \sqcup B' \in \mathcal{J}_\Lambda\}} \mathcal{C}_{\Lambda,B,B'} \tag{3.36}$$

Moreover, for each $S_1 \subset S$ we have

$$C_{\Lambda,B,S} = \bigsqcup_{S_2 \subset S_1} C_{\Lambda,B \sqcup S_2, S \setminus S_1}, \tag{3.37}$$

$$\mathcal{C}_{\Lambda,B,B',S} = \bigsqcup_{S_2 \subset S_1} \mathcal{C}_{\Lambda,B \sqcup S_2, B' \sqcup (S_1 \setminus S_2), S \setminus S_1}. \tag{3.38}$$

Lemma 3.7 leads to the following definition.

Definition 3.11. *Consider an energy $E \in \mathbb{R}$, $m > 0$, and a box $\Lambda = \Lambda_L(x)$. The Λ-bevent $\mathcal{C}_{\Lambda,B,B',S}$ and the Λ-bconfset $C_{\Lambda,B,S}$ are (Λ, E, m)-good if the box Λ is (B, S, E, m)-good.* (Note that Λ is then (ω, E, m)-jgood for every $\omega \in \mathcal{C}_{\Lambda,B,B',S}$.) *Those (Λ, E, m)-good bevents and bconfsets that are also Λ-dense will be called (Λ, E, m)-adapted.*

Definition 3.12. *Consider an energy $E \in \mathbb{R}$, a rate of decay $m > 0$, and a box Λ. We call Ω_Λ a (Λ, E, m)-localized event if there exist disjoint (Λ, E, m)-adapted bevents $\{\mathcal{C}_{\Lambda,B_i,B_i',S_i}\}_{i=1,2,\ldots,I}$ such that*

$$\Omega_\Lambda = \bigsqcup_{i=1}^{I} \mathcal{C}_{\Lambda,B_i,B_i',S_i}. \tag{3.39}$$

If Ω_Λ is a (Λ, E, m)-localized event, note that $\Omega_\Lambda \subset \Omega_\Lambda^{(0)}$ by its definition, and hence, recalling (3.38) and (3.33) , we can rewrite Ω_Λ in the form

$$\Omega_\Lambda = \bigsqcup_{j=1}^{J} \mathcal{C}_{\Lambda,A_j,A_j'}, \tag{3.40}$$

where the $\{\mathcal{C}_{\Lambda,A_j,A_j'}\}_{j=1,2,\ldots,J}$ are disjoint (Λ, E, m)-good bevents.

We will need (Λ, E, m)-localized events of scale appropriate probability.

Definition 3.13. *Fix $p > 0$. Given an energy $E \in \mathbb{R}$ and a rate of decay $m > 0$, a scale L is (E, m)-localizing if for some box $\Lambda = \Lambda_L$ (and hence for all) we have a (Λ, E, m)-localized event Ω_Λ such that*

$$\mathbb{P}\{\Omega_\Lambda\} > 1 - L^{-p}. \tag{3.41}$$

4. "A priori" finite volume estimates

Given an energy E, to start the multiscale analysis we will need, as in [B, BK], an *a priori* estimate on the probability that a box Λ_L is good with an adequate supply of free sites, for some sufficiently large scale L. The multiscale analysis will then show that such a probabilistic estimate also holds at all large scales.

Proposition 4.1. *Let $H_{\mathbf{X}}$ be an attractive Poisson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ with density $\varrho > 0$, and fix $p > 0$. Then there exists a scale $\overline{L}_0 = \overline{L}_0(d, u, \varrho, p) < \infty$, such that for all scales $L \geq \overline{L}_0$, setting*

$$\delta_L := (\varrho^{-1}(p + d + 1)\log L)^{\frac{1}{d}}, \quad E_L := -2u_+\varrho \log L, \tag{4.1}$$

and

$$m_{L,E} := \tfrac{1}{2}\sqrt{\tfrac{1}{2}E_L - E} \leq m_L := \tfrac{1}{2}\sqrt{-\tfrac{1}{2}E_L} \quad \text{for all } E \in]-\infty, E_L], \tag{4.2}$$

the scale L is $(E, m_{L,E})$-localizing for all energies $E \in]-\infty, E_L]$.

Proof. Let $\Lambda = \Lambda_L(x)$, and let δ_L and E_L be as in (4.1). If $Y \in \mathcal{Q}_\Lambda^{(0)}$, it follows from (3.18) and the Combes-Thomas estimate (e.g., [GK2, Eq. (19)]) that for all $E \in]-\infty, E_L]$ and all $t_Y \in [0,1]^Y$ we have, with $m_L(E)$ as in (4.2), that

$$\|R_{(Y,t_Y),\Lambda}(E)\| \leq (2m_{L,E})^{-2} \tag{4.3}$$

and

$$\|\chi_y R_{(Y,t_Y),\Lambda}(E)\chi_{y'}\| \leq \tfrac{1}{2}m_{L,E}^{-2}\,\mathrm{e}^{-2m_{L,E}|y-y'|} \text{ for } y, y' \in \Lambda \text{ with } |y - y'| \geq 4\sqrt{d}. \tag{4.4}$$

We now require $L > \delta_L + \delta_+$, and set

$$J := \{j \in x + \delta_L\mathbb{Z}^d \cap \Lambda;\ \Lambda_{\delta_L}(j) \subset \widehat{\Lambda}\}, \tag{4.5}$$

$$\widehat{\mathcal{J}}_\Lambda := \{S \in \mathcal{J}_\Lambda;\ N_S(\Lambda_{\delta_L}(j)) \geq 1 \text{ for all } j \in J\}. \tag{4.6}$$

If $S \in \widehat{\mathcal{J}}_\Lambda$, the density condition (3.29) for S in Λ follows from (4.6), and it follows from (4.3) and (4.4) that $\mathcal{C}_{\Lambda,\emptyset,\emptyset,S}$ is a $(\Lambda, E, m_{L,E})$-adapted bevent for all $E \in]-\infty, E_L]$ if $L \geq \overline{L}_1(d,u,\varrho,p)$. We conclude that

$$\Omega_\Lambda = \bigsqcup_{S\in\widehat{\mathcal{J}}_\Lambda} \mathcal{C}_{\Lambda,\emptyset,\emptyset,S} = \bigsqcup_{S\in\widehat{\mathcal{J}}_\Lambda} \{\mathbf{Y} \in [S]_\Lambda\} \tag{4.7}$$

is a $(\Lambda, E, m_{L,E})$-localizing event for all $E \in]-\infty, E_L]$.

To establish (3.41), let $\delta'_L := \frac{1}{2^d}\delta_L$ and consider the event

$$\Omega^{(\ddagger)}_\Lambda := \{N_{\mathbf{Y}}(\Lambda_{\delta'_L}(j)) \geq 1 \quad \text{for all} \quad j \in J\}. \tag{4.8}$$

We have

$$\mathbb{P}\{\Omega^{(\ddagger)}_\Lambda\} \geq 1 - \left(\tfrac{L}{\delta_L}\right)^d \mathrm{e}^{-2\varrho(\delta'_L)^d} = 1 - \frac{\varrho}{(p+d+1)L^{p+1}\log L} \geq 1 - \frac{1}{L^{p+1}}, \tag{4.9}$$

if $L \geq \mathrm{e}^{\frac{\varrho}{p+d+1}}$. Since $\Omega_\Lambda \supset \Omega^{(\ddagger)}_\Lambda \cap \Omega^{(0)}_\Lambda$, (3.41) follows from (4.9) and (3.19) for $L \geq \overline{L}_0(d,u,\varrho,p)$. □

5. The multiscale analysis and the proof of localization

The Bourgain-Kenig multiscale analysis, namely [BK, Proposition A′], was adapted to Poisson Hamiltonians in [GHK2, Proposition 5.1]. To apply the latter to attractive Poisson Hamiltonians we must show that the requirements of this multiscale analysis are satisfied. More precisely, we must show that attractive Poisson Hamiltonians satisfy appropriate versions of Properties SLI (Simon-Lieb inequality), EDI (eigenfunction decay inequality), IAD (independence at a distance), NE (average number of eigenvalues), and GEE (generalized eigenfunction expansion); see [GK1]. The Wegner estimate is proved by the multiscale analysis; it is not an "a priori requirement".

Since events based on disjoint boxes are independent, we have Property IAD. Property GEE is satisfied in view of (2.4), and we also have (2.5), which is needed in the multiscale analysis.

But Properties SLI, EDI and NE require some care and modification. In a box Λ_L we always work with Λ_L-acceptable configurations X, whence the potential V_{X,Λ_L} satisfies the lower bound (3.17). Inside the box Λ_L, Properties SLI and EDI (see [GK4, Theorem A.1], [BK, Section 2]) are governed by the same constant $\gamma_{E,L}$ given in [GK4, Eq. (A.2)], and hence for Λ_L-acceptable configurations we have

$$\gamma_{]-\infty,0],L} := \sup_{E\in]-\infty,0],L} \gamma_{E,L} \leq C_d\sqrt{u_+ \varrho \log L}. \tag{5.1}$$

For Property NE, it follows by the argument in [GK4, Eqs. (A.6)-(A.7)] that for all Λ_L-acceptable configurations X and energies $E \in]-\infty, 0]$ we have

$$\begin{aligned} \operatorname{tr}\left\{\chi_{(-\infty,E)}(H_{X,\Lambda_L})\right\} &\leq \operatorname{tr}\left\{\chi_{(-\infty,E+u_+\varrho\log L)}(-\Delta_{\Lambda_L})\right\} \\ &\leq C_d\,(u_+\varrho\log L)^{\frac{d}{2}}\,L^d. \end{aligned} \tag{5.2}$$

The extra factors of $\sqrt{\log L}$ in (5.1) and (5.2) are acceptable in the multiscale analysis.

The Wegner estimate is proved at each scale using [BK, Lemma 5.1′]. The sign of the single-site potential does not matter in this argument, as long as the single-site potential has a definite sign, positive or negative, to ensure the monotonicity of the eigenvalues in the free sites couplings.

Thus the following proposition follows from Proposition 4.1 and [GHK2, Proposition 5.1].

Proposition 5.1. *Let $H_{\mathbf{X}}$ be an attractive Poisson Hamiltonian on* $\mathrm{L}^2(\mathbb{R}^d)$ *with density $\varrho > 0$ and $p = \frac{3}{8}d-$. Then there exist an energy $E_0 = E_0(\varrho) < 0$ and a scale $L_0 = L_0(\varrho)$, such that setting $m_E := \frac{1}{4}\sqrt{\frac{1}{2}E_0 - E} \le m_{E_0} := \frac{1}{4}\sqrt{-\frac{1}{2}E_0}$, the scale L is (E, m_E)-localizing for all $L \ge L_0$ and $E \in]-\infty, E_0]$.*

Theorem 1.1 now follows from Proposition 5.1 as in [GHK2, Proposition 6.1].

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