# Fractal dimensions and the Phenomenon of Intermittency in Quantum Dynamics

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**Abstract**

We exhibit an intermittency phenomenon in quantum dynamics. More precisely we derive new lower bounds for the moments of order $p$ associated to the state $\psi(t) = \mathrm{e}^{-itH}\psi$ and averaged in time between 0 and $T$. These lower bounds are expressed in terms of generalized fractal dimensions $D^{\pm}_{\mu_\psi}(1/(1+p/d))$ of the measure $\mu_\psi$ (where $d$ is the space dimension). This improves previous results, obtained in terms of Hausdorff and Packing dimension.

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## 1 Introduction

A by now wide number of articles deals with the links between the quantum dynamics of wave packet solutions of the Schrödinger equation, and the spectral properties of the associated Hamiltonian $H$. Actually, during the last decade, an analysis originated by Guarneri in [14] and refined by others [5, 7, 22, 15, 25] established that the fractal properties of the spectral measures were relevant for the study of the spreading of wave packet. Consider a separable Hilbert space $\mathcal{H}$, an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$, and a self-adjoint operator $H$ on $\mathcal{H}$. Let $\psi_t = \mathrm{e}^{-itH}\psi$ be the solution of the Schrödinger equation

$$\left\{ \begin{array}{rcl} i\frac{\partial\psi_t}{\partial t} & = & H\psi_t \\ \psi_{t=0} & = & \psi \end{array} \right.$$

For $X = \sum_n n\langle .\ e_n\rangle\, e_n$ being the "position operator", we define the time averaged moments of order $p$ for $\psi$ as

$$\langle\langle |X|^p\rangle\rangle_{\psi,T} := \frac{1}{T}\int_0^T \left\| |X|^{p/2} \mathrm{e}^{-itH}\psi \right\|_{\mathcal{H}}^2 \mathrm{d}t = \frac{1}{T}\int_0^T \sum_{n\in\mathbb{N}} |n|^p |\langle \psi_t,\, e_n\rangle|^2 \mathrm{d}t. \tag{1.1}$$

In the specific case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, more relevant from a physical point of view, $\{e_n\}$ is the canonical basis $\{\delta_n\}_{n\in\mathbb{Z}^d}$ and $\langle\langle |X|^p\rangle\rangle_{\psi,T} = \frac{1}{T}\int_0^T \sum_{n\in\mathbb{Z}^d} |n|^p |\psi_t(n)|^2 \mathrm{d}t$. It is now well known from a series of results [1],[6],[7],[14],[25], in the case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ (extendable to $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d)$) that if $\dim_{\mathrm{H}}(\mu_\psi)$ is the Hausdorff dimension of the spectral measure $\mu_\psi$, then

$$\alpha^-(\psi,p,d) := \liminf_{T\to\infty} \frac{\log\langle\langle |X|^p\rangle\rangle_{\psi,T}}{\log T} \geq \dim_{\mathrm{H}}(\mu_\psi)\, \mathrm{p/d}\ . \tag{1.2}$$

More recently, in [15] a lower bound has been proven for the upper oscillations of $\langle\langle |X|^p\rangle\rangle_{\psi,T}$, namely

$$\alpha^+(\psi,p,d) := \limsup_{T\to\infty} \frac{\log\langle\langle |X|^p\rangle\rangle_{\psi,T}}{\log T} \geq \dim_{\mathrm{P}}(\mu_\psi)\, \mathrm{p/d}\ , \tag{1.3}$$

where $\dim_{\mathrm{P}}(\mu_\psi)$ is the "packing" dimension of $\mu_\psi$.

However those results are certainly not optimal. Some one-dimensional quantum systems with pure point spectrum can give rise to an almost ballistic motion [11], that is $\alpha^+(\psi,2,1) = 2$, meanwhile $\dim_{\mathrm{P}}(\mu_\psi) = \dim_{\mathrm{H}}(\mu_\psi) = 0$ for pure point measures. A similar phenomenon has been argued to hold for the random dimer model [12] [10]. In quasi-periodic models almost ballistic motion ($\alpha^+(\psi,2,1) = 2$) turns out to be a common phenomenon, actually a generic phenomenon [9], even in presence of purely zero Hausdorff dimensionality of the spectral measures [25, 9]. These examples show how far we are from a complete understanding of "What determines the spreading of a wave packet" [21]. In this paper, we try to go one step further in this undertanding, and also supply (Appendix B) a new enlightment concerning the main technique used in this field for the past ten years.

We obtain new lower bounds for the growth exponents of $\langle\langle |X|^p\rangle\rangle_{\psi,T}$, namely (1.4)-(1.5) below. As in (1.2)-(1.3) these bounds only rely on the fractal properties of the spectral measure $\mu_\psi$. We point out right now that unlike the existing results, the bounds (1.4)-(1.5) we get can be non trivial in the presence of zero dimensionality of the spectral measure ($\dim_{\mathrm{P}}(\mu_\psi) = \dim_{\mathrm{H}}(\mu_\psi) = 0$), even in the case of pure point spectrum (see Appendix D). Moreover in Appendix D, Theorem D.1, we show that there is no hope to improve our result by only taking into account the fractal properties of the spectral measure encoded in its generalized fractal dimensions $D^{\pm}_{\mu_\psi}(q)$.

Our result also provides a precise statement of a phenomenon discovered by recent numerical computations in some quantum models for quasicrystals [26, 29, 32]; this phenomenon has been called "Intermittency". Namely, it has been suggested by physicists that the growth exponents $\alpha^{\pm}(\psi,p,d)$ should grow faster than linearly in $p/d$ as proposed in (1.2) and (1.3); one should observe a more complex law for the behaviour of $\alpha^{\pm}(\psi,p,d)$ in the variable $p/d$: $\alpha^{\pm}(\psi,p,d) = \beta^{\pm}(\psi,p/d)p/d$, where $\beta^{\pm}(\psi,p/d)$ are non decreasing functions of $p/d$ (and of course non smaller than $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{P}}$).

These recent numerical investigations emphasized, in this phenomenon of intermittency, the role of more refined fractal quantities, the so-called "$q$-th generalized fractal dimensions"

$D^{\pm}_{\mu_\psi}(q)$. The lower bounds (1.4)-(1.5) that we establish for $\langle\langle |X|^p \rangle\rangle_{\psi,T}$ actually appear to be "intermittent" lower bounds, providing thus a rigourous statement concerning intermittency in quantum dynamics. More precisely, when $D^{\pm}_{\mu}(q)$ is non constant for $q \in (0,1)$, these lower bounds grow faster than linearly in $p/d$. After Theorem 2.1 we discuss the application of our result to an Hamiltonian constructed with Julia matrices with self-similar spectra [4][16]. For "real" Schrödinger operators, good candidates would be operators on $\ell^2(\mathbb{N})$ or $\ell^2(\mathbb{Z})$ with sparse or quasiperiodic (*e.g.* generated by substitution sequences) potential. But a careful analysis of the links between spectral properties and behaviour of the eigenfunctions would then be required; an analysis that goes, for instance, beyond the scope of [20] .

Our main result (Theorem 2.1) holds for any self-adjoint operators $H$, and for any initial state $\psi$ such that the associated spectral measure $\mathrm{d}\mu_\psi$ satisfies

$$(\mathbf{H}) \quad D^{\pm}_{\mu_\psi}(s) < +\infty \ \text{ for any } s \in (0,1).$$

The result reads then as follows:

$$\alpha^-(\psi,p,d) \geq D^-_{\mu_\psi}\left(\frac{1}{1+p/d}\right) p/d \tag{1.4}$$

and

$$\alpha^+(\psi,p,d) \geq D^+_{\mu_\psi}\left(\frac{1}{1+p/d}\right) p/d. \tag{1.5}$$

where $D^-_{\mu_\psi}(q)$ and $D^+_{\mu_\psi}(q)$ are the lower and upper "$q$-th generalized fractal dimensions" (see Definition 2.2). In Appendix C we discuss the validity of Hypothesis ($\mathbf{H}$), which, we note, holds for compactly supported measures. In particular Theorem 2.1 applies to the examples where the intermittency phenomenon has been argued to hold.

To achieve this we first derive a $p/d$-dependent lower bound $L_\psi(T)$ for $\langle\langle |X|^p \rangle\rangle_{\psi,T}$ (Theorem 3.1). Then we establish *via* Theorem 4.2 the connection between $L_\psi(T)$ and the generalized fractal dimensions, by adjusting for each single $T$ the "thin" part of $\mu_\psi$ that supplies the faster dynamical travel. Concerning the connection between $L_\psi(T)$ and $D^{\pm}_{\mu_\psi}(q)$ we moreover obtain a kind of optimality, in the sense that this quantity $L_\psi(T)$ that minors $\langle\langle |X|^p \rangle\rangle_{\psi,T}$ is shown to have its growth exponents exactly equal to $D^{\pm}_{\mu_\psi}(1/(1+p/d))\, p/d$.

We point out that $D^{\pm}_{\mu_\psi}(1/(1+p/d))$ are increasing functions of $p/d$, and are respectively not smaller than $\dim_{\mathrm{H}}(\mu_\psi)$ and $\dim_{\mathrm{P}}(\mu_\psi)$ (for all $p/d > 0$). Therefore (1.4)-(1.5) do improve the bounds (1.2) and (1.3) above.

These new bounds are a consequence of a double improvement of the approaches of Guarneri-Combes-Last (G-C-L) [7, 25] and of Barbaroux-Tcheremchantsev (BT) [5]. The first improvement is due, after a decompostion $\psi = \varphi + \chi$, $\varphi \perp \chi$, to a better control of the key quantity

$$\begin{aligned} B_\psi(T,N) \ &:= \ \frac{1}{T}\int_0^\infty \sum_{|n|\leq N} |\langle \mathrm{e}^{-itH}\psi, e_n\rangle|^2 h(t/T)\mathrm{d}t \\ &\leq \ B_\chi(T,N) + \frac{2}{T}\,\mathrm{Re}\int_0^\infty \sum_{|n|\leq N} \langle \mathrm{e}^{-itH}\varphi, e_n\rangle \overline{\langle \mathrm{e}^{-itH}\psi, e_n\rangle} h(t/T)\mathrm{d}t, \end{aligned} \tag{1.6}$$

and more particulary of the last term (the crossed term). We stress right now that this better control of the crossed term is essential, meaning that using the former available estimates

[25, 5] will not lead to the right fractal dimensions $D^{\pm}_{\mu_\psi}(1/(1+\beta))$, as illustrated in Appendix B. Here $h(z)$ is some positive function in $C_0^\infty([0,1])$ such that $\int_0^1 h(z)\mathrm{d}z = 1$.

Afterwards, and this is Theorem 3.1, one is able to obtain a constant $C(\psi,p,h) > 0$, such that for all $T > 0$,

$$\langle\langle |X|^p \rangle\rangle_{\psi,T} \geq C(\psi,p,h) L_\psi(T), \quad \text{with} \quad L_\psi(T) = \sup_{\varphi\in\mathcal{H}_\psi} \left\{ \frac{|\langle\varphi,\psi\rangle|^{2+2\beta}}{\|\varphi\|^2\,(U_{\varphi,\psi}(T))^\beta} \right\},$$

where $\mathcal{H}_\psi$ is the cyclic subspace spanned by $\psi$ and $H$,

$$U_{\varphi,\psi}(T) = \int_{\mathbb{R}^2} \mathrm{d}\mu_\psi(x)\mathrm{d}\mu_\varphi(y) R(T(x-y)),$$

and $R$ is some bounded fast decaying function defined by (3.5) in Section 3 (one should think to $R(w)$ as to the gaussian $e^{-w^2/4}$). Thus, as in [5, 15], one can "choose", for each $T$, a $T$-dependent vector $\varphi$ in the decomposition $\psi = \varphi+\chi$, that contains enough spectral information to approximate the supremum in $L_\psi(T)$. The second improvement then consists in the way one chooses this particular vector $\varphi$. We show (Theorem 4.1) that a judicious choice enables one to connect, up to a logarithmic factor, the quantity $L_\psi(T)$ to the integral

$$I_{\mu_\psi}\left(\frac{1}{1+p/d}, T^{-1}\right) = \int_{\mathbb{R}} \mathrm{d}\mu_\psi(x)\ \mu_\psi\left([x-T^{-1}, x+T^{-1}]\right)^{-\frac{p/d}{1+p/d}}$$

that defines the generalized fractal exponents $D^{\pm}_{\mu_\psi}(1/(1+p/d))$.

Our method also applies to the previous approaches (G-C-L) and (BT), where the crossed term in (1.6) wasn't treated well. It respectively yields, as stated in Appendix B, the fractal dimensions $D^{\pm}_{\mu_\psi}((1+\beta)/(1+2\beta))$ and $D^{\pm}_{\mu_\psi}((1+2\beta)/(1+3\beta))$. Since the functions $D^{\pm}_{\mu_\psi}(q)$ are non increasing functions of $q$, our Theorem 2.1 gives a better lower bound. This means that a better estimate of the crossed term in (1.6) does provide an improvement (Theorem B.3).

Finally, using extra assumptions on the decay of the generalized eigenfunctions $u(n,x)$ of $H$ (in the spirit of [21] [23]), it is possible to improve the above bounds. In particular, suppose that there exists a constant $C$ such that for $\mu_\psi$ a.e.$x$, $\sum_{|n|<N} |u(n,x)|^2 \leq CN^d$ holds, then $\alpha^{\pm}(\psi,p,d) \geq D^{\pm}_{\mu_\psi}(1-p/d)$ (Theorem 4.3).

The results of the present article together with a flavor of the proof of Theorem 4.1 can be found in the short note [2]. In [3] we deal with the continuous case.

While diffusing our short announcement [2], a related but weaker result by Guarneri and Schulz-Baldes was also released [17]. They obtained their result independently of us, using a quite different method. In particular they need a Large Deviation Theorem, a tool that does not enter in our proof. This enables us to obtain a more general result.

The paper is organized as follows: in Section 2 we define the generalized fractal dimensions $D^{\pm}_{\mu}(q)$ and state our main result, *i.e.* Theorem 2.1. The next two sections are devoted to its proof. In Section 3 we derive an abstract lower bound for $\langle\langle |X|^p \rangle\rangle_{\psi,T}$, that is $L_\psi(T)$ (Theorem 3.1), and in Section 4 we relate this quantity $L_\psi(T)$ to the generalized fractal

dimensions we defined (Theorems 4.1 and 4.2). Most of the results of Section 4 are proven under the assumption (**H**): $D^{\pm}_{\mu_\psi}(s) < +\infty$ for any $s \in (0,1)$.
In Appendix A we give the proof of statements ii) and iii) of Proposition 2.1.
In Appendix B we provide the analog of Theorems 4.1 and 4.2, for two others lower bounds (corresponding to the former approaches), and discuss their relations to $L_\psi(T)$.
In Appendix C we give a sufficient condition for the Hypothesis (H) to hold which can be useful in applications. In particular, (H) is true for any measure with compact support.
Finally, in Appendix D we give a simple example of pure point probability measure $\mu$ on $[0,1]$ which has strictly positive fractal dimensions $D^{\pm}_{\mu}(q)$ for some values of $q \in (0,1)$. And then we derive in Theorem D.1 an example of an Hamiltonian $H$ for which Theorem 2.1 is optimal.

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## 2 Definition of fractal dimensions and main result

**Definition 2.1.** *The Hausdorff and packing dimensions of a measure $\mu$ are respectively defined as (e.g. [15] and ref. therein)*

$$\dim_{\mathrm{H}}(\mu) = \mu - \operatorname*{ess\,sup}_{\mathrm{x}\in\mathbb{R}} \gamma_\mu^-(\mathrm{x}) \quad \text{and} \quad \dim_{\mathrm{P}}(\mu) = \mu - \operatorname*{ess\,sup}_{\mathrm{x}\in\mathbb{R}} \gamma_\mu^+(\mathrm{x}) \tag{2.1}$$

*where*

$$\gamma_\mu^-(x) = \liminf_{\varepsilon\to 0} \frac{\log\left(\mu([x-\varepsilon, x+\varepsilon])\right)}{\log\varepsilon} \quad \textit{and} \quad \gamma_\mu^+(x) = \limsup_{\varepsilon\to 0} \frac{\log\left(\mu([x-\varepsilon, x+\varepsilon])\right)}{\log\varepsilon},$$

*for $x \in \operatorname{supp}\mu$, and $\gamma_\mu^-(x) = \gamma_\mu^+(x) = +\infty$ for $x \notin \operatorname{supp}\mu$. The essential supremum is defined in the following way: take a set of full $\mu$-measure and compute the supremum over this set, and then take the infimum over all these sets of full $\mu$-measure.*

**Remark 2.1.** Note that the above definitions, while different from the "usual" ones ([28], [31]), are indeed equivalent to them: see *e.g.* [13] [15].

**Definition 2.2.** *Generalized fractal dimensions of a measure [19],[8].*
*Let $\mu$ be a (positive) Borelian probability measure. Let $q \in (-\infty, 1)$ and $\varepsilon \in (0,1)$. We consider the following function with values in $[1,\infty]$*

$$I_\mu(q,\varepsilon) = \int_{\mathrm{supp}\mu} \mu([x-\varepsilon, x+\varepsilon])^{q-1} \mathrm{d}\mu(x).$$

*The lower and upper generalized fractal dimensions of $\mu$ are respectively defined as*

$$D_\mu^-(q) = \frac{1}{1-q} \liminf_{\varepsilon\to 0} \frac{\log I_\mu(q,\varepsilon)}{-\log\varepsilon} \quad \textit{and} \quad D_\mu^+(q) = \frac{1}{1-q} \limsup_{\varepsilon\to 0} \frac{\log I_\mu(q,\varepsilon)}{-\log\varepsilon}. \tag{2.2}$$

*with the understanding that both are $+\infty$ if, for some $\varepsilon > 0$, $I_\mu$ takes the value $+\infty$.*

**Remark 2.2.**
i) For our purpose, it is sufficient to discuss the case $q \in (-\infty, 1)$ (see e.g. [8] for the general case).
ii) There exists actually a wide number of "generalized fractal dimensions". For example, they can be defined with the help of the so-called "singularity spectrum function" $f_\mu$ of the measure $\mu$ ([18], [27]), or as a solution of an implicit equation ([18, Formula 2.8], [27]). The resulting dimensions coincide with each other in certain very specific cases, like e.g. Cookie Cutter measures in $\mathbb{R}$ [27].

In order to state our results we also define the following integrals that could be considered as approximations of the quantities $I_\mu(q, \varepsilon)$. The function $R$ is a bounded even function with fast decay properties at $\pm\infty$, and will be precisely defined in (3.5) below.

$$K_\mu(q, \varepsilon) = \int_{\operatorname{supp}\mu} \mathrm{d}\mu(x) \left( \int_{\mathbb{R}} \mathrm{d}\mu(y) R((x-y)/\varepsilon) \right)^{q-1},$$

In Lemma 4.3 we shall prove that for any measure $\mu$ verifying the condition (H) of our theorem, taking $K_\mu(q, \varepsilon)$ in (2.2) instead of $I_\mu(q, \varepsilon)$ leads to the same values for the generalized fractal dimensions.

We review below some properties of the fractal dimension numbers $D_\mu^\pm(q)$ that will be of interest for us.

**Proposition 2.1.** *Let $\mu$ be a Borel probability measure.*
*i) $D_\mu^-(q)$ and $D_\mu^+(q)$ are non increasing functions of $q \in (-\infty, 1)$.*
*ii) For all $q \in (-\infty, 1)$, $D_\mu^-(q) \geq \dim_{\mathrm{H}}(\mu)$ .*
*iii) For all $q \in (-\infty, 1)$, $D_\mu^+(q) \geq \dim_{\mathrm{P}}(\mu)$.*
*iv) If $\mu$ has a bounded support, then for all $q \in (0, 1)$, $0 \leq D_\mu^-(q) \leq D_\mu^+(q) \leq 1$*

**Proof of Proposition 2.1.**
Statement i) is already known (see *e.g.* [8]). This a straightforward consequence of concave and convex Jensen inequalities. For the proof of ii), iii) see Appendix A. The statement iv) follows from Corollary C.1. Note that iv) does not necessarily hold any more if one lets $q$ vary in $(-\infty, 1)$ - See Appendix D. □

From now on, each time we will refer to the exponent $\beta$ it should be understood that $\beta = p$ if $\mathcal{H}$ is a general separable Hilbert space with basis $\{e_n\}_{n\in\mathbb{N}}$, and $\beta = p/d$ in the specific case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ equipped with its canonical basis $\{\delta_n\}_{n\in\mathbb{Z}^d}$.

Our main result is the following one:

**Theorem 2.1.** *Let $H$ be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$, and let $\psi$ be a vector in $\mathcal{H}$, $\|\psi\|_{\mathcal{H}} = 1$. Assume that the spectral measure $\mu_\psi$ associated to $\psi$ is such that*

$$\text{(H)} \quad D_{\mu_\psi}^\pm(s) < +\infty \quad \text{for any } s \in (0, 1).$$

*Then for $\langle\langle |X|^p \rangle\rangle_{\psi,T}$ defined by (1.1) and with $\beta$ as described just above, the following holds:*

$$\liminf_{T\to\infty} \frac{\log\langle\langle |X|^p \rangle\rangle_{\psi,T}}{\log T} \geq \beta D_{\mu_\psi}^- \left( \frac{1}{1+\beta} \right),$$

*and*

$$\limsup_{T\to\infty} \frac{\log\langle\langle |X|^p\rangle\rangle_{\psi,T}}{\log T} \geq \beta D^+_{\mu_\psi}\left(\frac{1}{1+\beta}\right).$$

**Remark 2.3.**
i) As a consequence of i)-iii) of Proposition 2.1, this result does improve previous known bounds of [25], [1],[15].
ii) As we show in Appendix D, Theorem D.1, for any $\beta = p/d > 0$ and any $\delta > 0$ there exist a bounded self-adjoint operator $H$ and a vector $\psi$ such that $\lim_{T\to\infty} \log\langle\langle |X|^p\rangle\rangle_{\psi,T}/\log T = 0$, but at the same time $D^\pm_{\mu_\psi}\left(\frac{1}{1+\beta} - \delta\right) > 0$. Therefore, one cannot hope to obtain a general (*i.e.* without additional assumptions on $\mu_\psi$ or on generalized eigenfunctions) lower bound of the form $\beta D^\pm_{\mu_\psi}(q(\beta))$ with some $q(\beta) < \frac{1}{1+\beta}$, *e.g.* like $D^\pm_{\mu_\psi}(1-\beta)$.

**Proof of Theorem 2.1.**
The proof of Theorem 2.1 is the combination of Theorem 3.1 and Theorem 4.2 which are proved respectively in Sections 3 and 4. □

We now discuss a model of Jacobi matrices, the Julia matrices. In this case, the upper bound derived in [4] together with Theorem 2.1 above enable one to prove for small $p$ that the increasing exponents of the moments of order $p$ are entirely controlled by the generalized fractal dimensions.

**Julia matrices:** There exists a class of models for which non trivial (i.e. non ballistic) upper bounds for the moments of order $p$ are derived in terms of generalized fractal dimensions: the Julia matrices. They are constructed by considering polynomials, disjoint Iterated Function Systems (IFS), giving rise to a real hyperbolic Julia set $J$ (see *e.g.* [4] [16] and references therein for details).

Given such an IFS, one considers the balanced measure $\mu$ of maximal entropy on $J$, and then constructs an Hamiltonian $H$ ($H$ corresponds to the Jacobi matrix associated to $\mu$) on $\ell^2(\mathbb{N})$ as follows.

Let $P_n$, $n \geq 0$, denote the orthogonal and normalized polynomials associated to $\mu$. The family $(P_n)_{n\in\mathbb{N}}$ forms a Hilbert basis in $\mathrm{L}^2(\mathbb{R},\mu)$ and satisfies a three terms recurrence relation $EP_n(E) = t_{n+1}P_{n+1}(E) + v_nP_n(E) + t_nP_{n-1}(E)$, $n \geq 0$, where $v_n \in \mathbb{R}$ and $t_n \geq 0$ are bounded sequences, and $P_{-1} = 0$. Therefore the isomorphism of $\mathrm{L}^2(\mathbb{R},\mu)$ onto $\ell^2(\mathbb{N})$ associated with the basis $(P_n)_{n\in\mathbb{N}}$ carries the operator of multiplication by $E$ in $\mathrm{L}^2(\mathbb{R},\mu)$ into the self-adjoint finite difference operator $H$ defined on $\ell^2(\mathbb{N})$ by

$$H\psi(n) = \begin{cases} t_{n+1}\psi(n+1) + t_n\psi(n-1) + v_n\psi(n) & n \geq 1 \\ t_1\psi(1) + v_0\psi(0) & n = 0 \end{cases} \tag{2.3}$$

Then $\mu$ is the spectral measure of $H$ associated to the state $\delta_0$ located at the origin. For this model, the upper and lower fractal dimensions $D^\pm_\mu(q)$ are equal ($:=D_\mu(q)$), and continuous for $q \in (0,1)$; furthermore, we have $D_\mu(1/(1+p)) = D_\mu(1-p) + \mathcal{O}(p^2)$ (see [4] and references therein).

It is established in [4] that there exists a critical value $p_c \geq 2$ such that for all $p \in (0, p_c)$

$$\alpha^+(\delta_0, p, 1) \leq D_\mu(1-p)$$

Therefore, putting together Theorem 2.1 and Theorem 1 of [4], we get, for the exponents of any moments of order $p \in (0, p_c)$ and for the initial state $\delta_0$, that

$$D_\mu\left(\frac{1}{1+p}\right) \le \alpha^{\pm}(\delta_0, p, 1) \le D_\mu(1-p) \ .$$

and thus for small $p > 0$,

$$D_\mu(1-p) + \mathcal{O}(p^2) \le \alpha^{\pm}(\delta_0, p, 1) \le D_\mu(1-p) \ .$$

This is the first model of Schrödinger-like operator treated rigorously for which such bounds are derived.

If $D_\mu(q)$ would be known to be strictly decreasing in some interval $(0, \delta)$, one would get intermittency for $\langle\langle |X|^p \rangle\rangle_\psi(T)$, with small $p$. However, to our best knowledge, this fact is only emphasized by numerics on the generalized fractal dimension (see *e.g.* [26]), and no rigourous results are provided.

$\square$

# 3 A general lower bound

This section can be regarded as the first part of the proof of our main result Theorem 2.1. Let $H$ be a self-adjoint operator in Hilbert space $\mathcal{H}$, and $\{e_n\}$ an orthonormal basis in $\mathcal{H}$ labelled by $n \in \mathbb{N}$ or by $n \in \mathbb{Z}^d$. Let $\psi$ be some vector in $\mathcal{H}$ such that $\|\psi\| = 1$. We are interested in lower bounds for the moments of the abstract position operator associated to the basis $\{e_n\}$, defined as

$$|X|^p_\psi(t) = \sum_n |n|^p |\langle \mathrm{e}^{-itH}\psi, e_n\rangle|^2. \tag{3.1}$$

In particular, if $\mathcal{H} = \ell^2(\mathbb{Z}^d)$, one can take the canonical basis $e_n(k) = \delta_{nk}, \quad n, k \in \mathbb{Z}^d$ to obtain the moments of the usual position operator. Our results can also be extended to the case $\mathcal{H} = \mathrm{L}^2(\mathbb{R}^d)$ [3] and

$$|X|^p_\psi(t) = \int_{\mathbb{R}^d} |x|^p |\psi(t,x)|^2 dx, \quad \psi(t,x) = (\mathrm{e}^{-itH}\psi)(x).$$

It can be done using the Theorem 3.2 in the same manner as in [5] (see Corollary 2.4 and Theorem 2.5 of [5], see also [25]). However, for the sake of simplicity, we shall consider in this paper only $|X|^p$ given by (3.1).

We derive a lower bound for the time averaged moments of position operator in term of an abstract quantity in the spirit of [5], namely

$$L_\psi(T) = \sup\left\{L_\psi(\varphi, T), \ \varphi \in \mathcal{H}_\psi, \ \ \langle\psi, \varphi\rangle \neq 0\right\}, \quad L_\psi(\varphi, T) = \frac{|\langle\varphi, \psi\rangle|^{2+2\beta}}{\|\varphi\|^2 U_{\varphi,\psi}(T)^\beta} \tag{3.2}$$

where $\mathcal{H}_\psi$ is the cyclic subspace spanned by $\psi$ and $H$. The exponent $\beta = p$ if $n \in \mathbb{N}$ and $\beta = p/d$ if $n \in \mathbb{Z}^d$. Notice that if $\varphi = \chi_\Omega(H)\psi$, where $\Omega$ is a Borel set, then $L_\psi(\varphi, T)$ reads as

$$L_\psi(\varphi, T) = \frac{\|\varphi\|^{2+4\beta}}{U_{\varphi,\psi}(T)^\beta}. \tag{3.3}$$

The quantity $U_{\varphi,\psi}(T)$ is defined as follows.

$$U_{\varphi,\psi}(T) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\psi(y) R((x-y)T), \tag{3.4}$$

where $R(w)$ is a bounded and fast decaying function defined in (3.5) below. In the sequel, we shall note ($\varepsilon$ playing the role of $1/T$),

$$b^{(R)}(x,\varepsilon) := \int_{\mathbb{R}} \mathrm{d}\mu_\psi(y) R\left(\frac{x-y}{\varepsilon}\right) .$$

So that $U_{\varphi,\psi}(T) = \int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x) b^{(R)}(x, 1/T)$. The quantity $U_{\varphi,\psi}(T)$ is crucial since one may consider that it codes the determinating part of the spectral informations that are involved in the dynamical behaviour of the considered quantum system. The quantity in (3.4) should be compared to other quantities such as

$$\int_{|x-y|\leq 1/T} \mathrm{d}\mu_\psi(x)\mathrm{d}\mu_\psi(y), \quad \text{or} \quad \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_\psi(x)\mathrm{d}\mu_\psi(y)\mathrm{e}^{-(x-y)^2T^2/4},$$

which already appear, in the limit $\varepsilon = T^{-1} \to 0$, as key quantities in order to discuss the nature of the spectrum [24],[25],[11],[5].

The main result of this section is the following.

**Theorem 3.1.** *Let $H$ be a self-adjoint operator on $\mathcal{H}$ and let $\psi$ be a vector in $\mathcal{H}$, $\|\psi\| = 1$. Let $\{e_n\}$ be some orthonormal basis in $\mathcal{H}$ and*

$$\langle\langle |X|^p \rangle\rangle_\psi(T) = \frac{1}{T}\int_0^T \sum_n |n|^p |\langle \mathrm{e}^{-itH}, e_n\rangle|^2 \mathrm{dt}.$$

*Let $h \in C_0^\infty([0,1])$ be any positive function such that $\int_0^1 h(z)dz = 1$ and define*

$$R(w) = \begin{cases} 1 & \text{if } |w| \leq 1 \\ |\hat{h}(w)|^2 & \text{if } |w| > 1, \end{cases} \tag{3.5}$$

*where $\hat{h}$ stands for the Fourier transform of $h$. Then, for all $p > 0$ and with $L_\psi(T)$ defined in (3.2), there exists a constant $C(\psi, p, h)$ such that for all $T > 0$*

$$\langle\langle |X|^p \rangle\rangle_\psi(T) \geq C(\psi, p, h) L_\psi(T) .$$

**Remark 3.1.** We point out that $h$ seems to be the necessary trick to take into account averaging on time between $[0, T]$ only, instead of $[-T, T]$. This trick actually allows one to deal with the crossed term in (1.6) and to recover a function $R$ with fast decay at $\pm\infty$. Replacing $h$ by the usual gaussian $e^{-z^2/4}$ in (3.15) below and thus in (3.6) will lead to the same result but with $|X|_\psi^p(t)$ averaged over $[-T, T]$.

As it is now well-known [14], [7], [25], a key point in the proof of lower bounds for $\langle\langle |X|^p\rangle\rangle_\psi(T)$ is a good control on the behaviour of the wave packet inside a ball of radius $N$, namely

$$B_\psi(T,N) = \frac{1}{T}\int_0^\infty \sum_{|n|\le N} |\langle \mathrm{e}^{-itH}\psi, e_n\rangle|^2 h(t/T)\mathrm{d}t. \tag{3.6}$$

To that end, we shall need Theorem 3.2 below, which is a generalization of Theorem 2.1 in [5]. In order to state it we first recall well-known facts involving the spectral theorem for the self-adjoint operator $H$ and the choosen vector $\psi$. (see *e.g.* [30] and ref. therein).
Namely, there exists a unitary map $W_\psi$ from the cyclic subspace $\mathcal{H}_\psi$ into the space $\mathrm{L}^2(\mathbb{R}, \mathrm{d}\mu_\psi)$ such that $W_\psi(\psi) = 1$ and $W_\psi(\mathrm{e}^{-itH}\psi) = \mathrm{e}^{-itx}$. We shall denote by $P_\psi$ the orthogonal projection on $\mathcal{H}_\psi$. The map $W_\psi$ has a kernel $u(n,x)$ defined by $u(n,\cdot) = W_\psi(P_\psi e_n)(\cdot)$, so that

$$\langle e^{-iHt}\psi, e_n\rangle = \int e^{-ixt}\overline{u(n,x)}\mathrm{d}\mu_\psi(x), \tag{3.7}$$

and more generally, for any vector $\xi \in \mathcal{H}$, one has

$$\langle P_\psi\xi, e_n\rangle = \int W_\psi\left(P_\psi\xi\right)(x)\overline{u(n,x)}\mathrm{d}\mu_\psi(x). \tag{3.8}$$

In the case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ and $e_n(k) = \delta_{nk}$, for each fixed $x \in \mathbb{R}$ the vector $(u(n,x)_{n\in\mathbb{Z}^d})$ may be seen as a generalized eigenfunction of $H$ (*i.e.* in a distributional sense). This observation is of interest for some applications ([24] and Theorem 4.3 below)

The expansions (3.7)-(3.8) are of course also possible with any vector $\varphi \in \mathcal{H}$ and corresponding kernel $v(n,y) = W_\varphi(P_\varphi e_n)(y)$ (actually if $\varphi = f(H)\psi$, with $f \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}\mu_\psi)$, then one checks that $v(n,y) = f(y)u(n,y)$).
We are now ready to formulate Theorem 3.2.

**Theorem 3.2.** *Let $h(z)$ be some function in $\mathrm{L}^1(\mathbb{R})$, $H$ a self-adjoint operator acting on $\mathcal{H}$ and $A$ a Hilbert-Schmidt operator in $\mathcal{H}$. For any couple of vectors $\psi, \varphi$ from $\mathcal{H}$ define the quantity*

$$D^{(h)}_{\varphi,\psi}(T) = \frac{1}{T}\int_{-\infty}^{+\infty} \langle A\mathrm{e}^{-itH}\varphi, \mathrm{e}^{-itH}\psi\rangle h(t/T)\mathrm{d}t.$$

*Let*

$$U^{(h)}_{\varphi,\psi}(T) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\psi(y)|\hat{h}((x-y)T)|^2.$$

*The following estimate holds:*

$$|D^{(h)}_{\varphi,\psi}(T)| \le \|A\|_2 \left(U^{(h)}_{\varphi,\psi}(T)\right)^{\frac{1}{2}},$$

*where $\|A\|_2$ is the Hilbert-Schmidt norm of $A$.*

*In the special case $A = \sum_{|n|\le N}\langle\cdot, e_n\rangle e_n$, one has $\|A\|_2 \le CN^{d/2}$, and therefore*

$$|D^{(h)}_{\varphi,\psi}(T,N)| = \left|\frac{1}{T}\int_{-\infty}^{+\infty} \sum_{|n|\le N} \langle \mathrm{e}^{-itH}\varphi, e_n\rangle\langle e_n, \mathrm{e}^{-itH}\psi\rangle h(t/T)\mathrm{d}t\right| \le CN^{d/2}\left(U^{(h)}_{\varphi,\psi}(T)\right)^{\frac{1}{2}}.$$

*Here $d = 1$ if one considers the abstract position operator associated with the base $\{e_n\}$ labelled by $n \in \mathbb{N}$, and $d \geq 1$ in the case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ equipped with the canonical basis $e_n = \delta_n$.*

**Proof of Theorem 3.2.**
Since $A$ is Hilbert-Schmidt, there exist two orthonormal bases $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$ and a monotonely decreasing sequence $\{E_n\}_{n\in\mathbb{N}}$, $E_n \geq 0$, such that $\sum_{n=1}^{\infty} E_n^2 = \|A\|_2^2 < +\infty$ and $A = \sum_{n=1}^{\infty} E_n \langle\cdot, f_n\rangle g_n$. Therefore,

$$D^{(h)}_{\varphi,\psi}(T) = \frac{1}{T}\int_{-\infty}^{+\infty}\sum_{n=1}^{\infty} E_n\langle \mathrm{e}^{-itH}\varphi, f_n\rangle\langle g_n, \mathrm{e}^{-itH}\psi\rangle h(t/T)\mathrm{d}t. \tag{3.9}$$

Then (3.7) reads as

$$\langle \mathrm{e}^{-itH}\varphi, f_n\rangle = \int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x)\mathrm{e}^{-itx}\overline{u(n,x)}\ , \tag{3.10}$$

where $u(n) = W_\varphi(P_\varphi f_n) \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}\mu_\varphi)$. The similar formula holds for $\psi$ with $v(n) = W_\psi(P_\psi g_n) \in \mathrm{L}^2(\mathbb{R}, \mathrm{d}\mu_\psi)$. One obtains from (3.9) -(3.10) that

$$D^{(h)}_{\varphi,\psi}(T) = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\psi(y)\hat{h}((x-y)T)S(x,y), \tag{3.11}$$

where

$$S(x,y) = \sum_{n=1}^{\infty} E_n\overline{u(n,x)}v(n,y).$$

The sum converges in $\mathrm{L}^2(\mathbb{R}^2, \mu_\varphi \times \mu_\psi)$. Applying the Cauchy-Schwarz inequality to (3.11), one gets

$$|D^{(h)}_{\varphi,\psi}(T)|^2 \leq U^{(h)}_{\varphi,\psi}(T)\|S\|^2_{\mathrm{L}^2(\mathbb{R}^2,\mathrm{d}\mu_\varphi\times\mathrm{d}\mu_\psi)}. \tag{3.12}$$

One can easily see that

$$\|S\|^2_{\mathrm{L}^2(\mathbb{R}^2,\mathrm{d}\mu_\varphi\times\mathrm{d}\mu_\psi)} = \sum_{n,k=1}^{\infty} E_nE_k a_{nk}\overline{b_{nk}},$$

where

$$a_{nk} = \int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x)u(k,x)\overline{u(n,x)} = \langle W_\varphi(P_\varphi f_k), W_\varphi(P_\varphi f_n)\rangle_{\mathrm{L}^2(\mathbb{R},\mathrm{d}\mu_\varphi)} = \langle P_\varphi f_k, f_n\rangle_{\mathcal{H}}\ , \tag{3.13}$$

where we used an analog of (3.8), and in the same manner

$$b_{nk} = \langle P_\psi g_k, P_\psi g_n\rangle_{\mathcal{H}} = \langle g_k, P_\psi g_n\rangle_{\mathcal{H}}.$$

We have also used the fact that both $P_\varphi$ and $P_\psi$ are orthogonal projections. By Parseval equality,

$$\sum_{n=1}^{\infty}|a_{nk}|^2 = \|P_\varphi f_k\|^2, \quad \sum_{k=1}^{\infty}|b_{nk}|^2 = \|P_\psi g_n\|^2.$$

Therefore, as $\|f_k\| = \|g_n\| = 1$ for all $k, n$,

$$\|S\|^4_{\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}\mu_\varphi \times \mathrm{d}\mu_\psi)} \le \sum_{k=1}^{\infty} E_k^2 \|P_\varphi f_k\|^2 \sum_{n=1}^{\infty} E_n^2 \|P_\psi g_n\|^2 \le \left( \sum_{n=1}^{\infty} E_n^2 \right)^2 = \|A\|_2^4. \tag{3.14}$$

The first statement of the Theorem follows from (3.12) and (3.14). The proof of the second part is essentially the same. The only difference is that in the case $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ the sums are taken over $n \in \mathbb{Z}^d : \ |n| \le N$. In particular, the estimate (3.14) reads as

$$\|S\|^4_{\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}\mu_\varphi \times \mathrm{d}\mu_\psi)} \le \sum_{|k| \le N} \|P_\varphi e_k\|^2 \sum_{|n| \le N} \|P_\psi e_n\|^2 \le C\, N^{2d}.$$

This ends the proof. $\square$

One should stress that the proof we presented here is simpler than the one of Theorem 2.1 in [5], because we do not use the product space $\mathcal{H} \otimes \mathcal{H}$. In the case $\psi = \varphi$ and $h(z) = \exp(-z^2/4)$ the result of Theorem 3.2 is equivalent to that of Theorem 2.1 in [5].

**Proof of Theorem 3.1.** Pick a positive function $h(z) \in C_0^\infty([0,1])$ such that $\int_0^1 h(z) \mathrm{d}z = 1$. The role of $h$ is to supply a fast decaying function $|\hat{h}(w)|^2$. Note that one trivially has, for any $z \in [0,1]$, $h(z) \le \|h\|_\infty \chi_{[0,1]}(z)$. Then one verifies that

$$\langle\langle X^p \rangle\rangle_\psi(T) \ge \frac{1}{\|h\|_\infty} \int_0^\infty \sum_n |n|^p |\langle \mathrm{e}^{-itH}\psi, e_n\rangle|^2 h(t/T) \frac{\mathrm{d}t}{T} \ge \frac{N^p}{\|h\|_\infty} \left( \|\psi\|^2 - B_\psi(T,N) \right), \tag{3.15}$$

with $B_\psi(T,N)$ defined line (3.6). As usual, one needs a control of this quantity $B_\psi(T,N)$ which represents the behaviour of the wave packet in a ball of radius $N$. Decompose the vector $\psi$ as $\varphi + \chi$, with $\langle \varphi, \chi \rangle = 0$ and $\varphi \neq 0$ (one should think to $\varphi = \chi_\Omega(H)\psi$). Thus

$$\begin{aligned} B_\psi(T,N) &= B_\varphi(T,N) + B_\chi(T,N) + \frac{2}{T} \mathrm{Re} \int_0^\infty \sum_{|n| \le N} \langle \mathrm{e}^{-itH}\varphi, e_n\rangle \overline{\langle \mathrm{e}^{-itH}\chi, e_n\rangle} h(t/T) \mathrm{d}t, \\ &= -B_\varphi(T,N) + B_\chi(T,N) + \frac{2}{T} \mathrm{Re} \int_0^\infty \sum_{|n| \le N} \langle \mathrm{e}^{-itH}\varphi, e_n\rangle \overline{\langle \mathrm{e}^{-itH}\psi, e_n\rangle} h(t/T) \mathrm{d}t. \end{aligned}$$

Then, taking into account that $1/T \int_0^\infty h(t/T)\mathrm{d}t = 1$ and $h(z) \ge 0$, we have $B_\chi(T,N) \le \|\chi\|^2 = \|\psi\|^2 - \|\varphi\|^2$. Let $A = \sum_{|n| \le N} \langle \cdot, e_n \rangle e_n$. Then

$$B_\psi(T,N) \le \|\psi\|^2 - \|\varphi\|^2 + 2\mathrm{Re} D^{(h)}_{\varphi,\psi}(T,N),$$

where $D^{(h)}_{\varphi,\psi}(T,N)$ was defined in the Theorem 3.2. The second statement of this Theorem gives immediately

$$B_\psi(T,N) \le \|\psi\|^2 - \|\varphi\|^2 + C N^{\frac{d}{2}} \left( U^{(h)}_{\varphi,\psi}(T) \right)^{\frac{1}{2}}, \tag{3.16}$$

where, as in Theorem 3.2,

$$U^{(h)}_{\varphi,\psi}(T) = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x) \mathrm{d}\mu_\psi(y) |\hat{h}((x-y)T)|^2.$$

As $|\hat{h}(w)| \leq 1$ for all $w$ and by definition (3.5) of $R$, we clearly have $R(w) \geq |\hat{h}(w)|^2$ for all $w$. Therefore $U^{(h)}_{\varphi,\psi}(T) \leq U_{\varphi,\psi}(T)$, and the estimate (3.16) is valid with $U_{\varphi,\psi}(T)$ defined by (3.4), that is with the function $R$ instead of $|\hat{h}|^2$.

We are now in position to finish the proof of Theorem 3.1. The basic strategy is standard: let $N$ be the largest integer such that $CN^{d/2}U_{\varphi,\psi}(T)^{\frac{1}{2}} \leq \|\varphi\|^2/2$, it yields:

$$B_\psi(T,N) \leq \|\psi\|^2 - \frac{\|\varphi\|^2}{2}. \tag{3.17}$$

The inequalities (3.17) and (3.15) yield with some positive constant $C(\psi,p,h)$:

$$\langle\langle |X|^p \rangle\rangle_{\psi,T} \geq C(\psi,p,h)\frac{\|\varphi\|^{2+4\beta}}{U_{\varphi,\psi}(T)^\beta} = C(\psi,p,h)L_\psi(\varphi,T), \quad \beta = \frac{p}{d}. \tag{3.18}$$

One recovers $L_\psi(\varphi,T)$ as given Line (3.3). It is this latter lower bound that will be used in the proof of the lower bound in Theorem 4.1. However, in the more general case where $\varphi$ is any function of $\mathcal{H}$ with $\langle\psi,\varphi\rangle \neq 0$, one gets the bound with

$$L_\psi(\varphi,T) := \frac{|\langle\varphi,\psi\rangle|^{2+2\beta}}{\|\varphi\|^2 U_{\varphi,\psi}(T)^\beta}. \tag{3.19}$$

Indeed take such a $\varphi$, and then define as in [5] $\tilde{\varphi} = (\langle\varphi,\psi\rangle\|\varphi\|^{-2})\varphi$; one thus checks that if $\chi = \psi - \tilde{\varphi}$, then $\langle\tilde{\varphi},\chi\rangle = 0$ and one is able to apply the result line (3.3) to $\psi$ and $\tilde{\varphi}$. Taking into account that $\|\tilde{\varphi}\| = |\langle\varphi,\psi\rangle|\|\varphi\|^{-1}$ and that $U_{\tilde{\varphi},\psi}(T) = |\langle\varphi,\psi\rangle|^2\|\varphi\|^{-4}U_{\varphi,\psi}(T)$, one finds the announced expression (3.19). To optimize the lower bound, we should take the supremum of $L_\psi(\varphi,T)$ for a given $T$ over all possible $\varphi$. One can show in the same manner as in [5], Lemma 3.1, that it is sufficient to take $\varphi$ only from the cyclic subspace $\mathcal{H}_\psi$. This gives us $L_\psi(T)$ defined line (3.2). □

# 4 Towards the fractal dimensions

This section deals with the connection between the dynamic quantity $L_\psi(T)$ introduced in the previous section, and the fractal dimensions defined in Section 2, called the generalized fractal dimensions.

We shall prove

**Theorem 4.1.** *Let $H$ be a self-adjoint operator on $\mathcal{H}$, and let $\psi$ be a vector in $\mathcal{H}$, $\|\psi\|_{\mathcal{H}} = 1$. Assume* **(H)***, namely the spectral measure $\mu_\psi$ associated to $\psi$ is such that*

$$D^{\pm}_{\mu_\psi}(s) < +\infty \quad \text{for any } s \in (0,1).$$

*Then, for all $\beta > 0$, there exists a constant $C_1 > 0$ such that, for all $\varepsilon > 0$:*

$$\frac{C_1}{|\log\varepsilon|^{1+\beta}} K_{\mu_\psi}\left(\frac{1}{1+\beta},\varepsilon\right)^{1+\beta} \leq L_\psi(\varepsilon^{-1}) \leq K_{\mu_\psi}\left(\frac{1}{1+\beta},\varepsilon\right)^{1+\beta}, \tag{4.1}$$

*where*

$$K_{\mu_\psi}\left(\frac{1}{1+\beta},\varepsilon\right) = \int_{\mathrm{supp}\mu_\psi} \mathrm{d}\mu_\psi(x)\left(\int_{\mathbb{R}} \mathrm{d}\mu_\psi(\mathrm{y})\mathrm{R}((\mathrm{x}-\mathrm{y})/\varepsilon)\right)^{\frac{-\beta}{1+\beta}}.$$

This will actually imply

**Theorem 4.2.** *Under the same hypothesis* (H) *as previously, one has*

$$\liminf_{T\to\infty}\frac{\log L_\psi(T)}{\log T}=\beta D^-_{\mu_\psi}\left(\frac{1}{1+\beta}\right),\quad \limsup_{T\to\infty}\frac{\log L_\psi(T)}{\log T}=\beta D^+_{\mu_\psi}\left(\frac{1}{1+\beta}\right).$$

**Remark 4.1.** i) We point out that the first inequality in (4.1), that is the lower bound on $L_\psi(\varepsilon^{-1})$, is sufficient to prove our main result Theorem 2.1. However the right side of (4.1), that is the upper bound, is of interest too. In particular it says that once one derived the lower bound $L_\psi(T)$, one cannot hope a better result than the one we stated. We shall also take advantage of the right part in (4.1) in Appendix B, Theorem B.3.
ii) Note that the proof of the right-hand-side inequality in (4.1), that is the upper bound, is true for any measures $\mu_\psi$.
iii) In Appendix C we show that (H) holds for all measures verifying the condition

$$\int_{\mathbb{R}}|x|^\lambda \mathrm{d}\mu_\psi(x)<+\infty \ \text{ for any } \lambda>0. \tag{4.2}$$

This is true, in particular, if $\mu_\psi$ has a compact support. Moreover, if (4.2) holds, $D^\pm_{\mu_\psi}(s)\in[0,1]$ for any $s\in(0,1)$.

We start by providing a proof for Theorem 4.1. Throughout this section, integrations must be systematically understood on the support of the measure $\mu_\psi$ or $\mu_\varphi$ we consider.

**Proof of the upper bound in Theorem 4.1.**

The upper bound of (4.1) in Theorem 4.1 will be proved, if taking any function $f\in \mathrm{L}^2(\mathbb{R},\mathrm{d}\mu_\psi)$, one shows

$$L_\psi(f(H)\psi,\varepsilon^{-1})\le K_{\mu_\psi}\left(\frac{1}{1+\beta},\varepsilon\right)^{1+\beta}=\left(\int \mathrm{d}\mu_\psi(\mathrm{x})\mathrm{b}^{(\mathrm{R})}(\mathrm{x},\varepsilon)^{\frac{-\beta}{1+\beta}}\right)^{1+\beta}, \tag{4.3}$$

where $b^{(R)}(x,\varepsilon)=\int \mathrm{d}\mu_\psi(y)R((x-y)/\varepsilon)$. This result will follow from Cauchy-Schwarz and Hölder Inequalities. Pick $\varphi=f(H)\psi$, with $f\in\mathrm{L}^2(\mathbb{R},\mathrm{d}\mu_\psi)$. Therefore

$$\mathrm{d}\mu_\varphi(x)=|f(x)|^2\mathrm{d}\mu_\psi(x) \text{ and } \langle\varphi,\psi\rangle=\int \mathrm{d}\mu_\psi(x)f(x).$$

Remember also that $U_{\varphi,\psi}(T)=\int \mathrm{d}\mu_\varphi(x)b^{(R)}(x,\varepsilon)$. Then, rewriting $L_\psi((f(H)\psi,\varepsilon^{-1})$ as given in (3.2) one gets :

$$L_\psi(f(H)\psi,\varepsilon^{-1})=\frac{\left|\int \mathrm{d}\mu_\psi(\mathrm{x})\mathrm{f}(\mathrm{x})\right|^{2+2\beta}}{\left(\int \mathrm{d}\mu_\psi(\mathrm{x})|\mathrm{f}(\mathrm{x})|^2\right)\left(\int \mathrm{d}\mu_\psi(\mathrm{x})|\mathrm{f}(\mathrm{x})|^2\mathrm{b}^{(\mathrm{R})}(\mathrm{x},\varepsilon)\right)^\beta}. \tag{4.4}$$

One starts with a Cauchy-Schwarz inequality applied to the numerator, and to the functions $b^{(R)}(x,\varepsilon)^{\frac{-\beta}{2(1+\beta)}}$ and $f(x)b^{(R)}(x,\varepsilon)^{\frac{\beta}{2(1+\beta)}}$. It yields:

$$\begin{aligned}\left|\int \mathrm{d}\mu_\psi(\mathrm{x})\mathrm{f}(\mathrm{x})\right|^{2+2\beta} &\le \left(\int \mathrm{d}\mu_\psi(\mathrm{x})\mathrm{b}^{(\mathrm{R})}(\mathrm{x},\varepsilon)^{\frac{-\beta}{1+\beta}}\right)^{1+\beta}\left(\int \mathrm{d}\mu_\psi(\mathrm{x})\mathrm{b}^{(\mathrm{R})}(\mathrm{x},\varepsilon)^{\frac{\beta}{1+\beta}}|\mathrm{f}(\mathrm{x})|^2\right)^{1+\beta} \\ &= K_{\mu_\psi}\left(\frac{1}{1+\beta},\varepsilon\right)^{1+\beta}\left(\int \mathrm{d}\mu_\psi(\mathrm{x})\mathrm{b}^{(\mathrm{R})}(\mathrm{x},\varepsilon)^{\frac{\beta}{1+\beta}}|\mathrm{f}(\mathrm{x})|^2\right)^{1+\beta}.\end{aligned} \tag{4.5}$$

An Hölder inequality applied to the last term, and with the coefficients $p = 1+\beta$ and $p' = (1+\beta)/\beta$, leads to:

$$\begin{aligned}
&\left(\int \mathrm{d}\mu_\psi(\mathrm{x})\mathrm{b}^{(\mathrm{R})}(\mathrm{x},\varepsilon)^{\frac{\beta}{1+\beta}}|\mathrm{f}(\mathrm{x})|^2\right)^{1+\beta} \\
&= \left(\int \mathrm{d}\mu_\psi(\mathrm{x})(|\mathrm{f}(\mathrm{x})|^2)^{\frac{1}{1+\beta}}\left(|\mathrm{f}(\mathrm{x})|^2\mathrm{b}^{(\mathrm{R})}(\mathrm{x},\varepsilon)\right)^{\frac{\beta}{1+\beta}}\right)^{1+\beta} \\
&\leq \left(\int \mathrm{d}\mu_\psi(\mathrm{x})|\mathrm{f}(\mathrm{x})|^2\right)\left(\int \mathrm{d}\mu_\psi(\mathrm{x})|\mathrm{f}(\mathrm{x})|^2\mathrm{b}^{(\mathrm{R})}(\mathrm{x},\varepsilon)\right)^{\beta}. \qquad (4.6)
\end{aligned}$$

One thus recovers exactly the denominator term, and (4.3) holds. □

**Remark 4.2.**
We stress the very strong link that comes out between $L_\psi(\varepsilon^{-1})$ and the integral $K_{\mu_\psi}(q,\varepsilon)$ with the particular value $q = 1/(1+\beta)$. The same appears to hold with the other lower bounds $L_1(T)$ and $L_2(T)$ defined in Appendix B and coming from the former approaches (G-C-L and BT). This shows how relevant are the fractal dimensions $D_\mu^\pm(q)$ with regards to the time behaviour of the quantities that have been studied for many years in quantum dynamics.

We now turn to the second and main part of Theorem 4.1, that is the lower bound. Our basic strategy to get the lower bound in (4.1) is to estimate the quantity $L_\psi(\varphi,T)$ in (3.19), with a vector $\varphi = \chi_{\Omega(r)}(H)\psi$. And $\Omega(r)$ will be a "thin" set of the form $\Omega(r) = \{x \in \mathrm{supp}\mu_\psi \mid \varepsilon^{r+A/N} < b^{(R)}(x,\varepsilon) \leq \varepsilon^r\}$, but which supports, roughly speaking, a significant part of the mass of the integral $K_{\mu_\psi}(1/(1+\beta),\varepsilon)$. The constant $A(q)$ is fixed by Lemma 4.1 below. The integer $N$ will stand for the integer part of $\log T$. Before going on with the proof of Theorem 4.1, we need the two following lemmas:

**Lemma 4.1.** *Let $q \in (0,1)$, and suppose that* (H) *holds for $\mu_\psi$, i.e. $D_{\mu\psi}^\pm(s) < +\infty$ for any $s \in (0,1)$. Define $b(x,\varepsilon) = \int \mathrm{d}\mu_\psi(x) g((x-y)\varepsilon^{-1})$, with $g(w) = \chi_{[-1,1]}(w)$ or $R(w)$. Then there exist $A = A(q)$ and $\varepsilon_0(q) > 0$ such that for all $\varepsilon \in (0,\,\varepsilon_0)$*

$$\int_{\{x\in\mathrm{supp}\mu \,\mid\, b(x,\varepsilon)\leq\varepsilon^A\}} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) \leq \varepsilon. \qquad (4.7)$$

**Lemma 4.2.** *Let $\mu_\psi$, $q$ and $A = A(q)$ be as in the previous lemma. Let $N > 0$ be an integer. Then, for $b(x,\varepsilon)$ defined as in Lemma 4.1, there exists an $r_0$ and a set $\Omega(r_0) = \{x \in \mathrm{supp}\mu_\psi \mid \varepsilon^{r_0+A/N} < b(x,\varepsilon) \leq \varepsilon^{r_0}\}$ such that, for all $\varepsilon$ small enough*

$$\int_{\Omega(r_0)} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) \geq \frac{1}{2N}\int b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x).$$

Lemma 4.2 is the key lemma to get Theorem 4.1, since it is this lemma that supplies the set $\Omega(r)$, and so the vector $\varphi = \chi_{\Omega(r)}(H)\psi$, that is needed to prove Theorem 4.1. Lemma 4.1 will not enter explicitely in the proof of Theorem 4.1 but is an important ingredient that we shall use twice: once while proving Lemma 4.2 and then in Lemma 4.3.

**Proof of the lower bound in Theorem 4.1.**

Let $N$ be the integer part of $-\log\varepsilon$. For a sake of simplicity we shall use $N = -\log\varepsilon$ (rather than the integer part). Take $A$ as in Lemma 4.1 and Lemma 4.2. Here $q = 1/(1+\beta)$, and thus $q-1 = -\beta/(1+\beta)$. For fixed $\varepsilon$, we choose $\varphi = \chi_{\Omega(r_0)}(H)\psi$ with $r_0$ given by Lemma 4.2. Thus, from the definition of $\Omega(r_0)$ we obtain

$$U_{\varphi,\psi}(\varepsilon^{-1}) = \int_{\Omega(r_0)} b^{(R)}(x,\varepsilon)\mathrm{d}\mu_\psi(x) \le \varepsilon^{r_0}\mu_\psi(\Omega(r_0)) \ ,$$

Therefore, using the expression (4.4) of $L_\psi(f(H)\psi,\varepsilon^{-1})$ with $f = \chi_{\Omega(r_0)}$, one has

$$\begin{aligned} L_\psi(\varepsilon^{-1}) \ &\ge\ \frac{\mu_\psi(\Omega(r_0))^{1+2\beta}}{\varepsilon^{r_0\beta}\mu_\psi(\Omega(r_0))^\beta} \ge \varepsilon^{\frac{A\beta}{N}}\left(\int_{\Omega(r_0)} b^{(R)}(x,\varepsilon)^{\frac{-\beta}{1+\beta}}\mathrm{d}\mu_\psi(x)\right)^{1+\beta} \\ &\ge\ \frac{e^{-A\beta}}{(-2\log\varepsilon)^{1+\beta}}\left(\int b^{(R)}(x,\varepsilon)^{\frac{-\beta}{1+\beta}}\mathrm{d}\mu_\psi(x)\right)^{1+\beta} = \frac{C_1}{|\log\varepsilon|^{1+\beta}}K_{\mu_\psi}\left(\frac{1}{1+\beta},\varepsilon\right)^{1+\beta}, \end{aligned} \tag{4.8}$$

where in the last inequality we used Lemma 4.2 and $N = -\log\varepsilon$. This holds for $\varepsilon$ small enough. $\square$

Remark that the fact that $r_0$ disappears in the relation (4.8) is crucial. It is of course related to the particular value $q = 1/(1+\beta)$ of the fractal dimension that enters into account. This is the place where the deep link between $L_\psi(T)$ and $D^{\pm}_{\mu_\psi}(1/(1+\beta))$ shows up.

We are left with the proofs of Lemmas 4.1 and 4.2.

**Proof of Lemma 4.1.**
For any $A > 0, \varepsilon > 0$ define

$$B(A,\varepsilon) = \{x \in \mathrm{supp}\mu \mid b(x,\varepsilon) \le \varepsilon^A\}.$$

Let $0 < s < q < 1$. As $b(x,\varepsilon) \ge \mu([x-\varepsilon, x+\varepsilon])$ whatever is $b(x,\varepsilon)$, we can estimate:

$$\begin{aligned} \int_{B(A,\varepsilon)} b(x,\varepsilon)^{q-1}\mathrm{d}\mu(\mathrm{x}) \ &=\ \int_{B(A,\varepsilon)} b(x,\varepsilon)^{q-s}b(x,\varepsilon)^{s-1}\mathrm{d}\mu(\mathrm{x}) \\ &\le\ \varepsilon^{A(q-s)}\int_{B(A,\varepsilon)} b(x,\varepsilon)^{s-1}\mathrm{d}\mu(\mathrm{x}) \\ &\le\ \varepsilon^{A(q-s)}\int_{B(A,\varepsilon)} \mu([x-\varepsilon,x+\varepsilon])^{s-1}\mathrm{d}\mu(\mathrm{x}) \\ &\le\ \varepsilon^{A(q-s)}I_\mu(s,\varepsilon). \end{aligned} \tag{4.9}$$

Let us take, for example, $s = q/2$. As $D^+_\mu(s) < +\infty$ for any $s > 0$, for $\varepsilon$ small enough one has

$$I_\mu(s,\varepsilon) \le \left(\frac{1}{\varepsilon}\right)^{(D^+_\mu(s)+1)(1-s)} .$$

Taking in (4.9) $A = (q-s)^{-1}((D^+_\mu(s)+1)(1-s)+1)$, we obtain the result of the Lemma. $\square$

**Proof of Lemma 4.2.**

To alleviate the notations, denote $B_A = \{x \in \mathrm{supp}\mu_\psi \mid b(x,\varepsilon) \le \varepsilon^A\}$ and also $B^A = \{x \in \mathrm{supp}\mu_\psi \mid b(x,\varepsilon) > \varepsilon^A\}$. Then, using the bound of Lemma 4.1, one has for $\varepsilon$ small enough

$$\begin{aligned}\int b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) &= \int_{B^A} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) + \int_{B_A} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x)\\ &\le \int_{B^A} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) + \varepsilon.\end{aligned}$$

Remark that one always has $b(x,\varepsilon) \le 1$ (since $R(w) \le 1$). Thus one can divide the remaining set $B^A$ into $N$ parts: $\Omega(kA/N) = \{x \in \mathrm{supp}\mu_\psi \mid \varepsilon^{(k+1)A/N} < b(x,\varepsilon) \le \varepsilon^{kA/N}\}$, with $k = 0, 1, \cdots, N-1$. At least one of these $N$ sets $\Omega(kA/N)$ gives rise to an integral bigger than $1/N$ times the integral over the whole set $B^A$. And so are picked $k_0$, $r_0 = Ak_0/N$, and thus the set $\Omega(r_0)$ of the Lemma.

To end the proof, remark that $\int b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) \ge 1$, since $q < 1$. Therefore, for $\varepsilon \le 1/2$, one gets

$$\begin{aligned}\int_{\Omega(r_0)} b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) &\ge \frac{1}{N}\left(\int b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x) - \varepsilon\right)\\ &\ge \frac{1}{2N}\int b(x,\varepsilon)^{q-1}\mathrm{d}\mu_\psi(x).\end{aligned}$$

□

We now turn to the proof of Theorem 4.2, which is actually a consequence of Theorem 4.1 and of the following Lemma 4.3 that relates the integrals $K_{\mu_\psi}(q,\varepsilon)$ to the integrals $I_{\mu_\psi}(q,\varepsilon)$ that enter into account in the definition of the generalized fractal dimension $D^{\pm}_{\mu_\psi}(q)$. More precisely, Lemma 4.3 says that, under Assumption **(H)** on $\mu_\psi$, both $K_{\mu_\psi}(q,\varepsilon)$ and $I_{\mu_\psi}(q,\varepsilon)$ have the same growth exponents $D^{\pm}_{\mu_\psi}(q)$. This is of course because of the fast decay properties of the function $R$ we have chosen (*via* the choice of $h \in C_0^\infty([0,1])$).

**Lemma 4.3.** *Let $q \in (0,1)$. Suppose that* (H) *holds for $\mu_\psi$. Then for all $\nu \in (0,1)$*

$$\frac{1}{2^{2-q}} I_{\mu_\psi}(q,\varepsilon^{1-\nu}) \le K_{\mu_\psi}(q,\varepsilon) \le I_{\mu_\psi}(q,\varepsilon), \tag{4.10}$$

*where the left inequality holds for $\varepsilon$ small enough ($\varepsilon \le \varepsilon(\nu)$). As a consequence*

$$\frac{1}{1-q}\liminf_{\varepsilon\to 0}\frac{\log K_{\mu_\psi}(q,\varepsilon)}{-\log\varepsilon} = D^-_{\mu_\psi}(q) \quad \text{and} \quad \frac{1}{1-q}\limsup_{\varepsilon\to 0}\frac{\log K_{\mu_\psi}(q,\varepsilon)}{-\log\varepsilon} = D^+_{\mu_\psi}(q). \tag{4.11}$$

**Remark 4.3.** We strongly believe that (4.11) holds in full generality for any $\mu_\psi$.

**Proof of Lemma 4.3.**

Throughout this proof we shall denote $B(x,\varepsilon) = [x-\varepsilon, x+\varepsilon]$. One has

$$I_{\mu_\psi}(q,\varepsilon) = \int \mu_\psi(B(x,\varepsilon))^{q-1}\mathrm{d}\mu_\psi(x) = \int \mathrm{d}\mu_\psi(x)\left(\int \chi_{[-1,1]}\left(\frac{x-y}{\varepsilon}\right)\mathrm{d}\mu_\psi(x)\right)^{q-1}.$$

Since, by (3.5), $\chi_{[-1,1]}(w) \leq R(w)$, and $q-1<0$, one has $K_{\mu_\psi}(q,\varepsilon) \leq I_{\mu_\psi}(q,\varepsilon)$. So we need to get the lower bound of (4.10). First notice that

$$\begin{aligned} b^{(R)}(x,\varepsilon) &= \int \mathrm{d}\mu_\psi(y) R\left(\frac{x-y}{\varepsilon}\right) \\ &= \int_{\left|\frac{x-y}{\varepsilon}\right|<\varepsilon^{-\nu}} \mathrm{d}\mu_\psi(y) R\left(\frac{x-y}{\varepsilon}\right) + \int_{\left|\frac{x-y}{\varepsilon}\right|\geq\varepsilon^{-\nu}} \mathrm{d}\mu_\psi(y) R\left(\frac{x-y}{\varepsilon}\right) \\ &\leq \mu_\psi(B(x,\varepsilon^{1-\nu})) + \sup_{|w|\geq\varepsilon^{-\nu}} R(w), \end{aligned}$$

where we used $R(w) \leq 1$. Let $\overline{R}(z) = \sup_{|w|\geq z} R(w)$. Define also

$$A_\nu^{(R)}(\varepsilon) = \{x \in \mathrm{supp}\mu_\psi \mid \mu_\psi(B(x,\varepsilon^{1-\nu})) \geq \overline{R}(\varepsilon^{-\nu})\} \text{ and } \tilde{A}_\nu^{(R)}(\varepsilon) = \left(A_\nu^{(R)}(\varepsilon)\right)^c \cap \mathrm{supp}\mu_\psi.$$

It is clear that for any $x \in A_\nu^{(R)}(\varepsilon)$, $b^{(R)}(x,\varepsilon) \leq 2\mu_\psi(B(x,\varepsilon^{1-\nu}))$.

Thus, with these notations, one has

$$\begin{aligned} K_{\mu_\psi}(q,\varepsilon) &= \int \mathrm{d}\mu_\psi(x) b^{(R)}(x,\varepsilon)^{q-1} \geq \int_{A_\nu^{(R)}(\varepsilon)} \mathrm{d}\mu_\psi(x) b^{(R)}(x,\varepsilon)^{q-1} \\ &\geq \int_{A_\nu^{(R)}(\varepsilon)} \mathrm{d}\mu_\psi(x) \left(2\mu_\psi(B(x,\varepsilon^{1-\nu}))\right)^{q-1} \\ &= \frac{1}{2^{1-q}}\left(I_{\mu_\psi}(q,\varepsilon^{1-\nu}) - \int_{\tilde{A}_\nu^{(R)}(\varepsilon)} \mathrm{d}\mu_\psi(x)\mu_\psi(B(x,\varepsilon^{1-\nu}))^{q-1}\right). \end{aligned}$$

Since $R(w)$ decays at $+\infty$ faster than any inverse power, for $\varepsilon$ small enough, that is $\varepsilon \leq \varepsilon(\nu)$, one has $\overline{R}(\varepsilon^{-\nu}) \leq \varepsilon^A$, where $A$ is the number from Lemma 4.1.

Hence, with $\tilde{A}_\nu(\varepsilon) = \{x \in \mathrm{supp}\mu_\psi \mid \mu_\psi(B(x,\varepsilon^{1-\nu})) \leq \varepsilon^A\}$,

$$\begin{aligned} 2^{1-q}K_{\mu_\psi}(q,\varepsilon) &\geq I_{\mu_\psi}(q,\varepsilon^{1-\nu}) - \int_{\tilde{A}_\nu(\varepsilon)} \mathrm{d}\mu_\psi(x)\mu_\psi(B(x,\varepsilon^{1-\nu}))^{q-1} && (4.12) \\ &\geq I_{\mu_\psi}(q,\varepsilon^{1-\nu}) - \varepsilon^{1-\nu} \geq \frac{1}{2} I_{\mu_\psi}(q,\varepsilon^{1-\nu}), && (4.13) \end{aligned}$$

if in addition $\varepsilon^{1-\nu} \leq 1/2$ (since $I_{\mu_\psi}(q,\varepsilon^{1-\nu}) \geq 1$). We have used Lemma 4.1 with $\varepsilon' = \varepsilon^{1-\nu}$ and with the function $g(w) = \chi_{[-1,1]}(w)$. This ends the proof of the first part. The conclusion of the lemma then follows: (4.13) yields

$$\frac{\log K_{\mu_\psi}(q,\varepsilon)}{-\log\varepsilon} \geq \frac{\log\left(2^{q-2} I_{\mu_\psi}(q,\varepsilon^{1-\nu})\right)}{-\log\varepsilon^{1-\nu}}(1-\nu),$$

for $\varepsilon$ small enough. Then take respectively the lim inf and lim sup and notice that the result is valid for all $\nu > 0$ as small as one wants. □

**Proof of Theorem 4.2.**

Theorem 4.2 is a direct consequence of Theorem 4.1 and of Lemma 4.3. □

As already mentioned at the end of Section 2, our main Theorem, namely Theorem 2.1, is a direct consequence of Theorem 3.1 (Section 3) and of Theorem 4.2.

We end this section with a few words concerning the case where, as in [21] and [23], one assumes some further properties on the spatial behaviour of the kernel $u(n, x)$. Assume that for some constant $C$, independent of the energy $x$, and for $\gamma \leq d$ one knows that for $\mu_\psi$ *a.e* $x$

$$\sum_{|n| \leq N} |u(n, x)|^2 \leq CN^\gamma \ , \qquad \text{(Hypothesis (HS))}$$

then one can not only get a better result taking $\beta = p/\gamma$ instead of $\beta = p/d$, but it turns out that one can push one step further our approach to reach the dimension numbers $D^{\pm}_{\mu_\psi}(1-\beta)$. The gain takes place in Inequality (3.12), where a direct estimate is made possible thanks to Hypothesis (HS). This thereby enables us to avoid the Cauchy-Schwarz inequality we made in order to split the integral into a spatial part and a spectral part. This therefore leads to the lower bound $\langle\langle |X|^p \rangle\rangle_{\psi,T} \geq CL'_\psi(T, \varphi) := C\frac{\|\varphi\|^{2+2\beta}}{U_{\varphi,\psi}(T)^\beta}$. Then the same technique as the one used to prove Theorem 4.1 supplies the following result.

**Theorem 4.3.** *Suppose in addition to the hypotheses of Theorem 2.1 that the spatial Hypothesis (HS) above holds for some $\gamma$ and constant $C$. Then, if $\beta = p/\gamma < 1$, one has*

$$\liminf_{T \to \infty} \frac{\log \langle\langle |X|^p \rangle\rangle_{\psi,T}}{\log T} \geq \liminf_{T \to \infty} \frac{\log L'_\psi(T)}{\log T} = \beta D^-_{\mu_\psi}(1-\beta),$$

*and*

$$\limsup_{T \to \infty} \frac{\log \langle\langle |X|^p \rangle\rangle_{\psi,T}}{\log T} \geq \limsup_{T \to \infty} \frac{\log L'_\psi(T)}{\log T} = \beta D^+_{\mu_\psi}(1-\beta).$$

# Appendices

## A Complement of Section 2

For the reader's convenience, we provide the proof of statements ii) and iii) of Proposition 2.1.

**Proof of ii) of Proposition 2.1.** We first note that from the convex Jensen Inequality, and since $q - 1 < 0$, we have, for any set $A \subset \mathbb{R}$

$$\begin{aligned} I_\mu(q, \varepsilon/2)^{\frac{1}{q-1}} &= \left( \int_{\mathbb{R}} \mu(x - \varepsilon/2, x + \varepsilon/2)^{q-1} \mathrm{d}\mu(x) \right)^{\frac{1}{q-1}} \\ &\leq \left( \int_A \mu(x - \varepsilon, x + \varepsilon)^{q-1} \mathrm{d}\mu(x) \right)^{\frac{1}{q-1}} \leq \int_A \mu(x - \varepsilon, x + \varepsilon) \mathrm{d}\mu(x) \ . \end{aligned} \tag{A.1}$$

Consider now, for all $\nu \in (0, 1)$, the set

$$A^{(H)}_\nu(\varepsilon) \equiv \{x \in \mathbb{R} \mid \mu(x - \varepsilon, x + \varepsilon) < \varepsilon^{\dim_{\mathrm{H}}(\mu) - \nu}\} \ .$$

We have

$$A_\nu^{(H)} := \liminf_{\varepsilon\to 0} A_\nu^{(H)}(\varepsilon) \supset \{x\in\mathbb{R} \mid \gamma_\mu^-(x) > \dim_{\mathrm{H}}(\mu)-\nu\}.$$

By the definition of $\dim_{\mathrm{H}}(\mu)$ given Line (2.1) we get $\mu(\{x\in\mathbb{R} \mid \gamma_\mu^-(x) > \dim_{\mathrm{H}}(\mu)-\nu\}) > 0$. Furthermore, $\liminf_{\varepsilon\to 0}\mu(A_\nu^{(H)})(\varepsilon) \geq \mu(A_\nu^{(H)}) \geq \mu(\{x\in\mathbb{R} \mid \gamma_\mu^-(x) > \dim_{\mathrm{H}}(\mu)-\nu\}) > 0$. Thus, letting $A = A_\nu^{(H)}(\varepsilon)$ in (A.1), we obtain

$$D_\mu^-(q) = \liminf_{\varepsilon\to 0}\frac{\log I_\mu(q,\varepsilon/2)^{\frac{1}{q-1}}}{\log(\varepsilon/2)} \geq$$

$$\liminf_{\varepsilon\to 0}\frac{\log\left(\mu(A_\nu^{(H)}(\varepsilon))\varepsilon^{\dim_{\mathrm{H}}(\mu)-\nu}\right)}{\log(\varepsilon/2)} = \dim_{\mathrm{H}}(\mu)-\nu\ .$$

Since the above inequality is valid for all $\nu\in(0,1)$, ii) of Proposition 2.1 is proven.

**Proof of iii) of Proposition 2.1.** We define, for $\varepsilon_k = \mathrm{e}^{-k}$

$$A_\nu^{(P)}(\varepsilon_k) := \{x\in\mathbb{R} \mid \mu(x-\varepsilon_k, x+\varepsilon_k) < \varepsilon_k^{\dim_{\mathrm{P}}(\mu)-\nu}\}\ .$$

Since $\lim_{k\to\infty}\log\varepsilon_k/\log\varepsilon_{k+1} = 1$, we have $\limsup_{k\to\infty}\log\mu(x-\varepsilon_k,x+\varepsilon_k)/\log\varepsilon_k = \limsup_{\varepsilon\to 0}\log\mu(x-\varepsilon,x+\varepsilon)/\log\varepsilon$. Thus we get

$$\begin{aligned} A_\nu^{(P)} &:= \limsup_{k\to\infty} A_\nu^{(P)}(\varepsilon_k) \\ &\supset \{x\in\mathbb{R} \mid \limsup_{k\to\infty}\frac{\mu(x-\varepsilon_k,x+\varepsilon_k)}{\log\varepsilon_k} > \dim_{\mathrm{P}}(\mu)-\nu\} \\ &= \{x\in\mathbb{R} \mid \gamma_\mu^+(x) > \dim_{\mathrm{P}}(\mu)-\nu\}. \end{aligned}$$

Therefore, from the definition of $\dim_{\mathrm{P}}(\mu)$ given Line (2.1) and the above inclusions, we get $\mu(A_\nu^{(P)}) > 0$. Using the Borel Cantelli lemma (as done in [15]) implies that $\sum_k \mu(A_\nu^{(P)}(\varepsilon_k)) = \infty$ and thus, there exists a subsequence $\varepsilon_{k(n)} \searrow 0$ of $\varepsilon_k$ such that $\mu(A_\nu^{(P)}(\varepsilon_{k(n)})) \geq k(n)^{-2} = (\log\varepsilon_{k(n)})^{-2}$. Thus, letting $A = A_\nu^{(P)}(\varepsilon_{k(n)})$ in (A.1), we obtain

$$D_\mu^+(q) = \limsup_{\varepsilon\to 0}\frac{\log I_\mu(q,\varepsilon)^{\frac{1}{q-1}}}{\log\varepsilon} \geq \lim_{n\to\infty}\frac{\log\left(\mu(A_\nu^{(P)}(\varepsilon_{k(n)}))\varepsilon_{k(n)}^{\dim_{\mathrm{P}}(\mu)-\nu}\right)}{\log\varepsilon_{k(n)}} = \dim_{\mathrm{P}}(\mu)-\nu\ .$$

Again, since the result is true for all $\nu\in(0,1)$, statement iii) of Proposition 2.1 is proven. □

# B Relation to other lower bounds

Getting a growth exponent in terms of fractal dimensions $D_{\mu_\psi}^\pm(q)$ is not specific to our lower bound $L_\psi(T)$ (3.2). It is also possible to get such relations from the lower bounds formerly derived, either directly (Barbaroux-Tcheremchantsev's lower bound $L_2(T)$ below), either improving them (Guarneri-Combes-Last's lower bound, improved by optimizing in $\varphi$ for each single $T$, see $L_1(T)$ below). We shall briefly explain this point in this section, and thereby propose to the reader a link between the present work and the former methods. In particular this appendix illustrates how actually deeply connected are the generalized fractal dimensions $D^{\pm}_{\mu_\psi}(q)$ to the lower bounds of $\langle\langle |X|^p\rangle\rangle_{\psi,T}$ studied for the past ten years, that is since Guarneri [14].

Since we shall focus on the relation between those lower bounds and the fractal dimensions $D^{\pm}_{\mu_\psi}(q)$, and for a sake of simplicity, we shall only consider vectors $\varphi$ of the form $\chi_\Omega(H)\psi$ instead of general $\varphi \in \mathcal{H}_\psi$ as in (3.2).

We first consider the lower bound that appears in [5]. For some constant $C(\psi,p) > 0$, it is shown that for all $T > 0$,

$$\langle\langle |X|^p\rangle\rangle_{\psi,T} \geq C(\psi,p)L_2(T), \quad \text{with} \quad L_2(T) = \sup\left\{L_2(\varphi,T) := \frac{\|\varphi\|^{2+8\beta}}{S_\varphi(T)^\beta},\ \varphi = \chi_\Omega(H)\psi\right\},$$

where

$$S_\varphi(T) = \int_{\mathbb{R}^2} \mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\varphi(y)R(T(x-y)).$$

Here $R(w)$ can be chosen as the usual gaussian $e^{-w^2/4}$. Then one can mimic the proof of Theorem 3.1, and relate the quantity $L_2(\varphi,T) = \|\varphi\|^{2+8\beta}/S_\varphi(T)^\beta$ to the integral $K_{\mu_\psi}(\frac{1+2\beta}{1+3\beta}, T^{-1})$. More precisely, the same kind of Hölder Inequalities as in (4.5)-(4.6) supplies an upper bound for $L_2(T)$ and using a set $\Omega(r)$ in the same spirit as previously yields the lower bound, again up to a logarithmic factor. The following theorem then holds:

**Theorem B.1.** *Under the same hypotheses as in Theorem 2.1, one has*

$$\liminf_{T\to\infty} \frac{\log L_2(T)}{\log T} = \beta D^-_{\mu_\psi}\left(\frac{1+2\beta}{1+3\beta}\right), \quad \text{and} \quad \limsup_{T\to\infty} \frac{\log L_2(T)}{\log T} = \beta D^+_{\mu_\psi}\left(\frac{1+2\beta}{1+3\beta}\right).$$

The second lower bound we want to discuss in this section is the improved version, using [5], of the first kind of quantity that has been considered in order to bound from below $\langle\langle |X|^p\rangle\rangle_{\psi,T}$, and that comes from Guarneri-Combes-Last [14, 7, 25]. Roughly, the basic idea is to take into account the function $G_\varphi(T) := \mu_\varphi - \operatorname{ess\,sup} \int \mathrm{d}\mu_\varphi(y)R(T(x-y))$ where $R(w) = \exp(-w^2/4)$ (notice that $G_\varphi(T)$ will be smaller than $CT^{-\alpha}$ if $\mu_\varphi$ is uniformly $\alpha$-Hölder continuous, see (3.10) in [25]). One then uses $G_\varphi(T)$ in order to bound $B_\varphi(T,N)$ defined as in (3.6) but with $1/T\int_0^T$ rather than $1/T\int_0^{+\infty} h(t/T)$. Using, for instance, $2|v(n,x)v(n,y)| \leq |v(n,x)|^2 + |v(n,y)|^2$,

$$\begin{aligned} B_\varphi(T,N) &\leq C\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}\mu_\varphi(x)\mathrm{d}\mu_\varphi(y)R((x-y)T)\sum_{|n|\leq N}|v(n,x)v(n,y)| \\ &\leq CG_\varphi(T)\int_{\mathbb{R}}\mathrm{d}\mu_\varphi(x)\sum_{|n|\leq N}|v(n,x)|^2 \leq C'G_\varphi(T)N^d. \end{aligned}$$

Following [25, 5], this leads to the lower bound

$$\langle\langle |X|^p\rangle\rangle_{\psi,T} \geq C(\psi,p)L_1(T), \quad \text{with} \quad L_1(T) = \sup\left\{L_1(\varphi,T) := \frac{\|\varphi\|^{2+4\beta}}{G_\varphi(T)^\beta},\ \varphi = \chi_\Omega(H)\psi\right\}.$$

One should compare the obtained lower bound $L_1(T)$ to the expression (6.10) in [25].

Then one can again mimic the proof of Theorem 4.1, and relate the quantity $L_1(T)$ to the integral $K_{\mu_\psi}(\frac{1+\beta}{1+2\beta}, T^{-1})$. This leads to the following theorem:

**Theorem B.2.** *Under the same hypotheses as in Theorem 2.1, one has*

$$\liminf_{T\to\infty} \frac{\log L_1(T)}{\log T} = \beta D^-_{\mu_\psi}\left(\frac{1+\beta}{1+2\beta}\right), \quad \text{and} \quad \limsup_{T\to\infty} \frac{\log L_1(T)}{\log T} = \beta D^+_{\mu_\psi}\left(\frac{1+\beta}{1+2\beta}\right).$$

One should also compare the expression of $L_1(T)$ and $L_2(T)$ to the expression of $L_\psi(T)$ given Line (3.3), that is where as above the supremum is taken over the set of vectors $\varphi = \chi_\Omega(H)\psi$, namely the supremum of $\|\varphi\|^{2+4\beta}/U_{\varphi,\psi}(T)^\beta$. One may have the right to wonder whether there are some links between these quantities, and also whether $L_\psi(T)$ is effectively a better lower bound than $L_1(T)$ and $L_2(T)$. Theorem B.3 below answers to these questions. This theorem is actually a consequence of Theorems 4.1, B.2 and B.1, and of the non increasing property of the functions $D^\pm_{\mu_\psi}(q)$ (point i) of Proposition 2.1).

**Theorem B.3.** *Under the same hypotheses as previously, one has*

$$\liminf_{T\to\infty} \frac{\log L_\psi(T)}{\log T} \geq \liminf_{T\to\infty} \frac{\log L_1(T)}{\log T} \geq \liminf_{T\to\infty} \frac{\log L_2(T)}{\log T},$$

*and*

$$\limsup_{T\to\infty} \frac{\log L_\psi(T)}{\log T} \geq \limsup_{T\to\infty} \frac{\log L_1(T)}{\log T} \geq \limsup_{T\to\infty} \frac{\log L_2(T)}{\log T}.$$

**Remark B.1.** i) It is worth to point out that such a comparison is made possible thanks to the upper bounds obtained for $L_\psi(T)$, $L_1(T)$ and $L_2(T)$; upper bounds that are for the first time derived for such quantities.
ii) Theorem B.3 tells us that it is worth to deal with the crossed term in (1.6) while one develops $B_\psi(T, N)$ with $\psi = \varphi + \chi$. Dealing with the crossed term does lead to an improvement, with regards to the former approaches (G-C-L) and (BT) where this term wasn't treated well.
iii) One can show that $L_\psi(\varphi, T) \geq c_1 L_1(\varphi, T)$ with some constant $c_1 > 0$ uniform in $T, \varphi$. The latter is not true as one compares $L_\psi(\varphi, T)$ and $L_2(\varphi, T)$ for some $\varphi$. We confess that the inequalities involving $L_2(T)$ in Theorem B.3 are quite a surprise for us, since we were expecting the exponents of $L_2(T)$ to be bigger than those of $L_1(T)$.
iv) We believe a stronger version of Theorem B.3 to be true, namely: $L_\psi(T) \geq c_1 L_1(T) \geq c_2 L_2(T)$.

# C A sufficient condition for Hypothesis (H)

The following statement gives a sufficient condition for (H) to hold, which can be useful for applications.

**Proposition C.1.** *Let* $0 < q < 1$. *Assume that,*

$$\textbf{(H1)} \quad \text{for some } \lambda > \frac{1-q}{q}, \text{ one has } \int_{\mathbf{R}} |x|^\lambda \mathrm{d}\mu(\mathrm{x}) < +\infty.$$

*Then* $0 \leq D^\pm_\mu(q) \leq 1$.

**Proof of Proposition C.1.**
Let $b(x,\varepsilon) = \mu([x-\varepsilon, x+\varepsilon])$. Define

$$\Omega(\varepsilon) = \{x \in \mathrm{supp}\mu \mid b(x,\varepsilon) \le \varepsilon\},$$

and $I_j = [(j-1/2)\varepsilon, (j+1/2)\varepsilon)$, $j \in \mathbb{Z}$. For any $x \in I_j$,

$$\mu([x-\varepsilon, x+\varepsilon]) \ge \mu(I_j) \ge \mu(I_j \cap \Omega(\varepsilon)) := b_j. \tag{C.1}$$

We first remark that for any $j$, $b_j \le \varepsilon$. Indeed, if $b_j = 0$, it is trivially true. And if $b_j > 0$, there exists $x_0 \in I_j \cap \Omega(\varepsilon)$. The inequality (C.1) and the definition of $\Omega(\varepsilon)$ imply

$$b_j \le \mu([x_0 - \varepsilon, x_0 + \varepsilon]) \le \varepsilon.$$

Next, the inequality (C.1) yields

$$\begin{aligned} \int_{\Omega(\varepsilon)} b(x,\varepsilon)^{q-1} \mathrm{d}\mu(x) &= \sum_{j:b_j>0} \int_{I_j\cap\Omega(\varepsilon)} b(x,\varepsilon)^{q-1}\mathrm{d}\mu(x) \\ &\le \sum_{j:b_j>0} \mu(I_j\cap\Omega(\varepsilon))^{q-1}\mu(I_j\cap\Omega(\varepsilon)) = \sum_{j\in\mathbb{Z}} b_j^q. \end{aligned} \tag{C.2}$$

One can rewrite this summation as follows:

$$\sum_{j\in\mathbb{Z}} b_j^q = \sum_{k=0}^{+\infty} S_k, \quad S_k := \sum_{j\in J_k} b_j^q, \tag{C.3}$$

where $J_k = \{j \in \mathbb{Z} \mid I_j \subset [e^k, e^{k+1}) \cup [-e^{k+1}, -e^k)\}$ for $k > 0$ and $J_0 = \{j \mid I_j \subset [-e, e)\}$. To be rigourous, Equality (C.3) holds if one replaces in the definition of the sets $J_k$ the quantity $e^k$ by $[e^k/\varepsilon]\varepsilon + \varepsilon/2$, where $[\cdot]$ stands for the integer part, but for a sake of simplicity we shall use the definition of $J_k$ given above.

Let $\gamma \in (0,1)$. For $k \ge 0$, Hölder inequality and the fact that $\mathrm{Card}J_k \le 2(e^{k+1} - e^k)/\varepsilon \le 4e^k/\varepsilon$ and $b_j \le \varepsilon$ imply

$$\begin{aligned} S_k &= \sum_{j\in J_k} b_j^q = \sum_{j\in J_k} b_j^{\gamma q} b_j^{(1-\gamma)q} \le \left(\sum_{j\in J_k} b_j\right)^{\gamma q} \left(\sum_{j\in J_k} b_j^{\frac{(1-\gamma)q}{1-\gamma q}}\right)^{1-\gamma q} \\ &\le \varepsilon^{(1-\gamma)q} \left(\frac{4e^k}{\varepsilon}\right)^{1-\gamma q} \left(\sum_{j\in J_k} b_j\right)^{\gamma q} = 4\varepsilon^{q-1} e^{k(1-\gamma q)} \left(\sum_{j\in J_k} b_j\right)^{\gamma q} \end{aligned} \tag{C.4}$$

From Assumption (H1) it is straightforward that there exists $C(\lambda) < \infty$ such that for all $k$

$$\sum_{j\in J_k} b_j \le \sum_{j\in J_k} \mu(I_j) \le C(\lambda)e^{-k\lambda}. \tag{C.5}$$

with $\lambda$ from the statement of the Proposition. Then one can find $\gamma \in (0,1)$ such that $\lambda\gamma q - (1-\gamma q) > 0$. Then (C.3)-(C.5) imply

$$\sum_j b_j^q \le 4C^{\gamma q}(\lambda)\varepsilon^{q-1} \sum_{k=0}^{+\infty} e^{-k(\lambda\gamma q - (1-\gamma q))} \le D(q,\lambda)\varepsilon^{q-1}. \tag{C.6}$$

The proof is completed as follows. One can write $I_\mu(q,\varepsilon)$ as

$$I_\mu(q,\varepsilon) = \left( \int_{\Omega(\varepsilon)} + \int_{\mathbb{R}\setminus\Omega(\varepsilon)} \right) b(x,\varepsilon)^{q-1} \mathrm{d}\mu(x) =: I_1 + I_2.$$

The bounds (C.2) and (C.6) give $I_1 \leq D\varepsilon^{q-1}$. Furthermore, due to the definition of $\Omega(\varepsilon)$ and $q \in (0,1)$, we have

$$I_2 \leq \varepsilon^{q-1} \mu(\mathbb{R} \setminus \Omega(\varepsilon)) \leq \varepsilon^{q-1}.$$

We obtain so $I_\mu(q,\varepsilon) \leq (D+1)\varepsilon^{q-1}$ and thus $D_\mu^\pm(q) \leq 1$.

□

**Corollary C.1.** *Let $\mu$ be such that*

$$\int_{\mathbb{R}} |x|^\lambda \mathrm{d}\mu(x) < +\infty \quad \text{for any } \lambda > 0.$$

*Then $D_\mu^\pm(s) \in [0,1]$ for any $s \in (0,1)$ and thus* (H) *holds. In particular, this is true if $\mu$ has a compact support.*

# D An example of finite pure point measure with non trivial generalized fractal dimensions.

For $\lambda > 1$ and $\alpha > 0$, let $a_n = a/n^\lambda$, $x_n = 1/n^\alpha$, where $a > 0$ is a normalization constant. We define the finite pure point probability measure $\mu = \sum_{n=1}^\infty a_n \delta_{x_n}$ on $\mathbb{R}$. As the measure $\mu$ has a bounded support, $0 \leq D_\mu^\pm(q) \leq 1$ for any $q \in (0,1)$ - see Point iv) of Proposition 2.1. We denote by $B(x,\varepsilon)$ the closed ball of center $x$ and radius $\varepsilon$. For any given $\varepsilon > 0$, take $N$ to be the integer part of $\varepsilon^{-1/(1+\alpha)}$. Thus, there exists a constant $c > 0$ uniform in $N$ (and $\varepsilon$) such that for all $n \leq cN$, $\mu(B(x_n,\varepsilon)) = \mu(\{x_n\}) = a_n$. Let $q < 1/\lambda$. Then we have

$$I_\mu(q,\varepsilon) = \sum_n a_n \mu(B(x_n,\varepsilon))^{q-1} \geq \sum_{n=1}^{cN} \mu(B(x_n,\varepsilon))^{q-1} a_n = \sum_{n=1}^{cN} a_n^q \sim C\, \varepsilon^{-\frac{1-q\lambda}{1+\alpha}} \,. \tag{D.1}$$

Therefore

$$D_\mu^\pm(q) \geq \frac{1-q\lambda}{(1-q)(1+\alpha)}.$$

This implies that for $0 < q < 1/\lambda$, one obtains strictly positive generalized fractal dimensions $D_\mu^\pm(q)$ for the pure point measure $\mu$. Moreover, by taking $q$ and $\alpha$ small, one can render these dimensions as close to 1 as one wants.

The estimate (D.1) is actually also valid for $q < 0$ (and one can show that the dimensions $D_\mu^\pm(q)$ are finite for any $q < 0$). However, the behaviour of the fractal dimensions is rather strange: they can be greater than 1 and the bigger is $\lambda$, the bigger are $D_\mu^\pm(q)$ as $q \to -\infty$:

$$\liminf_{q\to-\infty} D_\mu^\pm(q) \geq \frac{\lambda}{1+\alpha} > 1,$$

if $\alpha$ is small enough. Furthermore note that if $\mu$ is now defined with $a_n = ae^{-\lambda n}$, then $D_\mu^\pm(q) = +\infty$ whenever $q < 0$. This shows that one should be very cautious when considering possible physical applications of $D_\mu^\pm(q)$ with $q < 0$.

To conclude, we give the example of a self-adjoint operator $H$, and a state $\psi$ as mentioned in Remark 2.3 ii). We shall prove:

**Theorem D.1.** *Let $\beta > 0$ and $\delta > 0$. There exist a bounded self-adjoint operator $H$ on $\mathcal{H} = \ell^2(\mathbb{N})$ and a state $\psi \in \mathcal{H}$ such that, with $p = \beta$,*

$$\lim_{T\to\infty} \frac{\log\langle\langle |X|^p\rangle\rangle_{\psi,T}}{\log T} = 0, \ \text{but} \ D_{\mu_\psi}^\pm\left(\frac{1}{1+\beta} - \delta\right) > 0.$$

**Proof of Theorem D.1**.
Let $\beta > 0$ and $\delta > 0$ be as in the theorem, and $\nu > 0$ to be chosen later on. Let $(e_n)_{n\geq 1}$ be the canonical basis of $\ell^2(\mathbb{N}^*)$: $e_n(k) = \delta_{nk}$, $k, n \geq 1$. Define a self-adjoint operator $H$ in $\mathcal{H}$ as follows:

$$He_n = x_n e_n, \quad x_n = n^{-\alpha}, \quad n \geq 1.$$

It is easy to see that the spectral measure $\mu_\psi$ associated to the vector $\psi(k) = \sqrt{a}k^{-\lambda/2}$, $\lambda = 1 + \beta + \nu$, is exactly the measure $\mu$ defined above (so $\mathrm{supp}\mu_\psi \subset [0, 1]$ and Theorem 2.1 applies). Notice that $\psi(t, k) = \exp(-itx_k)\psi(k)$. Then, as $d = 1$ one has $\beta = p$, and one checks that

$$\langle\psi_t, |X|^p\psi_t\rangle = \sum_{k\geq 1} ak^{\beta-\lambda} = C(\nu) < +\infty.$$

Therefore, $\lim_{T\to\infty} \log\langle\langle |X|^p\rangle\rangle_{\psi,T}/\log T = 0$. On the other hand, as we have seen above, $D_{\mu_\psi}^\pm(q) > 0$ for any $q < 1/\lambda = \frac{1}{1+\beta+\nu}$. Taking $\nu$ small enough, *e.g.* such that $\frac{1}{1+\beta+\nu} = \frac{1}{1+\beta} - \delta/2$, one gets the required example. $\square$

**Remark D.1.** on can easily construct such an example on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ too, by numbering the canonical basis $(e_n)_{n\in\mathbb{Z}^d}$ in a "spiral" way, and taking $\beta = p/d$ as usual. Indeed, one then gets $e_n(k) = 1$ for some $k \sim n^{1/d}$.

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