LOCALIZATION FOR SCHRÖDINGER OPERATORS WITH POISSON RANDOM POTENTIAL

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Abstract. We prove exponential and dynamical localization for the Schrödinger operator with a nonnegative Poisson random potential at the bottom of the spectrum in any dimension. We also conclude that the eigenvalues in that spectral region of localization have finite multiplicity. We prove similar localization results in a prescribed energy interval at the bottom of the spectrum provided the density of the Poisson process is large enough.

Contents

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1. Introduction and main results

Consider an electron moving in an amorphous medium with randomly placed identical impurities, each impurity creating a local potential. For a fixed configuration of the impurities, described by the countable set $X \subset \mathbb{R}^d$ giving their locations, this motion is described by the Schrödinger equation $-i\partial_t \psi_t = H_X \psi_t$ with the Hamiltonian

$$H_X := -\Delta + V_X \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^d), \tag{1.1}$$

where the potential is given by

$$V_X(x) := \sum_{\zeta \in X} u(x - \zeta), \tag{1.2}$$

with $u(x-\zeta)$ being the single-site potential created by the impurity placed at ζ. Since the impurities are randomly distributed, the configuration X is a random countable subset of $\mathbb{R}^d$, and hence it is modeled by a point process on $\mathbb{R}^d$. Physical considerations usually dictate that the process is homogeneous and ergodic with respect to the translations by $\mathbb{R}^d$, cf. the discussions in [LiGP, PF]. The canonical point process with the desired properties is the homogeneous Poisson point process on $\mathbb{R}^d$.

The *Poisson Hamiltonian* is the random Schrödinger operator $H_{\mathbf{X}}$ in (1.1) with $\mathbf{X}$ a Poisson process on $\mathbb{R}^d$ with density $\varrho > 0$. The potential $V_{\mathbf{X}}$ is then a *Poisson random potential*. Poisson Hamiltonians may be the most natural random Schrödinger operators in the continuum as the distribution of impurities in a various samples of material is naturally modeled by a Poisson process. A mathematical proof of the existence of localization has been a long-standing open problem (cf. the survey [LMW]). The Poisson Hamiltonian has been long known to have Lifshitz tails [DV, CL, PF, Klo3, Sz, KloP, St1], a strong indication of localization at the bottom of the spectrum. Up to now localization had been shown only in one dimension [Sto], where it holds at all energies, as expected.

In this article we prove localization for nonnegative Poisson Hamiltonians at the bottom of the spectrum in arbitrary dimension. We obtain both exponential (or Anderson) localization and dynamical localization, as well as finite multiplicity of eigenvalues.

The Poisson Hamiltonian $H_{\mathbf{X}}$ is an $\mathbb{R}^d$-ergodic family of random self-adjoint operators. It follows from standard results (cf. [KiM, PF]) that there exists fixed subsets of $\mathbb{R}$ so that the spectrum of $H_{\mathbf{X}}$, as well as the pure point, absolutely continuous, and singular continuous components, are equal to these fixed sets with probability one.

In the multi-dimensional continuum case, there are a wealth of results concerning localization for Anderson-type Hamiltonians. These are $\mathbb{Z}^d$-ergodic random Schrödinger operators as in (1.1) but for which the location of the impurities is fixed at the vertices of the lattice $\mathbb{Z}^d$ (i.e., $X \equiv \mathbb{Z}^d$), and the single-site potentials are multiplied by random variables with bounded densities, e.g., [HM, CoH, Klo2, KiSS, Klo4, GK3, AENSS]. Localization was shown for a $\mathbb{Z}^d$-ergodic random displacement model where the displacement probability distribution has a bounded density [Klo1]. In contrast, a lot less is known about $\mathbb{R}^d$-ergodic random Schrödinger operators (random amorphous media). There are localization results for a class of Gaussian random potentials [FiLM, U, LMW]. Localization for

Poisson models where the single-site potentials are multiplied by random variables with bounded densities has also been studied [MS, CoH]. What all these results have in common is the availability of random variables with densities which can be exploited, in an averaging procedure, to produce an *a priori* Wegner estimate at all scales (e.g., [HM, CoH, Klo2, CoHM, Ki, FiLM, CoHN, CoHKN, CoHK]).

But if these random variables with densities (or Holder continuous distributions [CKM, St2]) are not available, as in the case of the Poisson Hamiltonians, it is a totally different story, and up to recently there were no localization results if $d \geq 2$.

This changed with Bourgain and Kenig's remarkable proof of localization for the Bernoulli-Anderson Hamiltonian, an Anderson-type Hamiltonian where the coefficients of the single-site potentials are Bernoulli random variables [BoK]. They established a Wegner estimate by a multiscale analysis using "free sites" and a new quantitative version of unique continuation which gives a lower bound on eigenfunctions. Since they obtained weak probability estimates and had discrete random variables, they also introduced a new method to prove Anderson localization from estimates on the finite-volume resolvents given by a single-energy multiscale analysis. The new method does not use the perturbation of singular spectra method nor Kotani's trick as in [CoH, SW], which requires random variables with bounded densities. It is also not an energy-interval multiscale analysis as in [DrK, FrMSS, Kl], which requires better probability estimates.

To prove localization for Poisson Hamiltonians we use the new ideas introduced by Bourgain and Kenig [Bo, BoK]. To apply these ideas, developed for the Bernoulli-Anderson Hamiltonian, in the case of the Poisson Hamiltonian, we exploit the probabilistic properties of Poisson point processes.

In this article the single-site potential u is a nonnegative, nonzero L^∞-function on $\mathbb{R}^d$ with compact support, with

$$u_- \chi_{\Lambda_{\delta_-}(0)} \leq u \leq u_+ \chi_{\Lambda_{\delta_+}(0)} \quad \text{for some constants } u_\pm, \delta_\pm \in]0, \infty[\tag{1.3}$$

where $\Lambda_L(x)$ denotes the box of side L centered at $x \in \mathbb{R}^d$. It follows that $\sigma(H_{\mathbf{X}}) = [0, +\infty[$ with probability one [KiM].

We need to introduce some notation. For a given set B, we denote by χ_B its characteristic function, by $\mathcal{P}_0(B)$ the collection of all countable subsets of B, and by $\#B$ its cardinality. Given $X \in \mathcal{P}_0(B)$ and $A \subset B$, we set $X_A := X \cap A$ and $N_X(A) := \#X_A$. Given a Borel set $A \subset \mathbb{R}^d$, we write $|A|$ for its Lebesgue measure. We let $\Lambda_L(x) := x + \left(-\frac{L}{2}, \frac{L}{2}\right)^d$ be the box of side L centered at $x \in \mathbb{R}^d$. By Λ we will always denote some box $\Lambda_L(x)$, with Λ_L denoting a box of side L. We set $\chi_x := \chi_{\Lambda_1}(x)$, the characteristic function of the box of side 1 centered at $x \in \mathbb{R}^d$. We write $\langle x \rangle := \sqrt{1 + |x|^2}$, $T(x) := \langle x \rangle^\nu$ for some fixed $\nu > \frac{d}{2}$. By $C_{a,b,\ldots}$, $K_{a,b,\ldots}$, etc., will always denote some finite constant depending only on $a, b, \ldots$.

A Poisson process on a Borel set $B \subset \mathbb{R}^d$ with density (or intensity) $\varrho > 0$ is a map $\mathbf{X}$ from a probability space $(\Omega, \mathbb{P})$ to $\mathcal{P}_0(B)$, such that for each Borel set $A \subset B$ with $|A| < \infty$ the random variable $N_{\mathbf{X}}(A)$ has Poisson distribution with mean $\varrho|A|$, i.e.,

$$\mathbb{P}\{N_{\mathbf{X}}(A) = k\} = \tfrac{(\varrho|A|)^k}{k!} \mathrm{e}^{-\varrho|A|} \quad \text{for } k = 0, 1, 2, \ldots, \tag{1.4}$$

and the random variables $\{N_{\mathbf{X}}(A_j)\}_{j=1}^n$ are independent for disjoint Borel subsets $\{A_j\}_{j=1}^n$ (e.g., [K, R]).

For Poisson random potentials the density ϱ is a measure of the amount of disorder in the medium. Our first result gives localization at fixed disorder at the bottom of the spectrum.

Theorem 1.1. *Let $H_{\mathbf{X}}$ be a Poisson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ with density $\varrho > 0$. Then there exist $E_0 = E_0(\varrho) > 0$ and $m = m(\rho) > 0$ for which the following holds $\mathbb{P}$-a.e.: The operator $H_{\mathbf{X}}$ has pure point spectrum in $[0, E_0]$ with exponentially localized eigenfunctions with rate of decay m, i.e., if ϕ is an eigenfunction of $H_{\mathbf{X}}$ with eigenvalue $E \in [0, E_0]$ we have*

$$\|\chi_x \phi\| \le C_{\mathbf{X},\phi}\, e^{-m|x|}, \quad \text{for all } x \in \mathbb{R}^d. \tag{1.5}$$

Moreover, there exist $\tau > 1$ and $s \in]0, 1[$ such that for all eigenfunctions ψ, ϕ (possibly equal) with the same eigenvalue $E \in [0, E_0]$ we have

$$\|\chi_x \psi\| \, \|\chi_y \phi\| \le C_{\mathbf{X}} \|T^{-1}\psi\| \|T^{-1}\phi\| \, e^{\langle y \rangle^\tau} e^{-|x-y|^s}, \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{1.6}$$

In particular, the eigenvalues of $H_{\mathbf{X}}$ in $[0, E_0]$ have finite multiplicity, and $H_{\mathbf{X}}$ exhibits dynamical localization in $[0, E_0]$, that is, for any $p > 0$ we have

$$\sup_t \| \langle x \rangle^p e^{-itH_{\mathbf{X}}} \chi_{[0,E_0]}(H_{\mathbf{X}}) \chi_0 \|_2^2 < \infty. \tag{1.7}$$

The next theorem gives localization at high disorder in a fixed interval at the bottom of the spectrum.

Theorem 1.2. *Let $H_{\mathbf{X}}$ be a Poisson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ with density $\varrho > 0$. Given $E_0 > 0$, there exist $\varrho_0 = \varrho_0(E_0) > 0$ and $m = m(E_0) > 0$ such that the conclusions of Theorem 1.1 hold in the interval $[0, E_0]$ if $\varrho > \varrho_0$.*

Theorems 1.1 and 1.2 are proved by a multiscale analysis as in [Bo, BoK], where the Wegner estimate, which gives control on the finite volume resolvent, is obtained by induction on the scale. In contrast, the usual proof of localization by a multiscale analysis [FrS, FrMSS, Sp, DrK, CoH, FK, GK1, Kl] uses an *a priori* Wegner estimate valid for all scales. Exponential localization will then follow from this new single-energy multiscale analysis as in [BoK, Section 7]. The decay of eigenfunction correlations exhibited in (1.6) follows from a detailed analysis of [BoK, Section 7] given in [GK5], using ideas from [GK4]. Dynamical localization and finite multiplicity of eigenvalues follow from (1.6). That (1.6) implies dynamical localization is rather immediate. The finite multiplicity of the eigenvalues follows by estimating $\|\chi_x \chi_{\{E\}}(H_{\mathbf{X}})\|_2^2 \|\chi_y \chi_{\{E\}}(H_{\mathbf{X}})\|_2^2$ from (1.6) and summing over $x \in \mathbb{Z}^d$.

Bourgain and Kenig's methods [BoK] were developed for the Bernoulli-Anderson Hamiltonian. Let $\boldsymbol{\varepsilon}_{\mathbb{Z}^d} = \{\boldsymbol{\varepsilon}_\zeta\}_{\zeta \in \mathbb{Z}^d}$ denote independent identically distributed Bernoulli random variables, $\boldsymbol{\varepsilon}_\zeta = 0$ or 1 with equal probability. The Bernoulli-Anderson random potential is $V(x) = \sum_{\zeta \in \mathbb{Z}^d} \boldsymbol{\varepsilon}_\zeta u(x - \zeta)$, and the Hamiltonian has the form (1.1). To see the connection with the Poisson Hamiltonian, let us introduce the Bernoulli-Poisson Hamiltonian. We consider a configuration $Y \in \mathcal{P}_0(\mathbb{R}^d)$, and let $\boldsymbol{\varepsilon}_Y = \{\boldsymbol{\varepsilon}_\zeta\}_{\zeta \in Y}$ be the corresponding collection of independent identically distributed Bernoulli random variables. We define the Bernoulli-Poisson Hamiltonian by $H_{(Y,\boldsymbol{\varepsilon}_Y)} := -\Delta + \sum_{\zeta \in Y} \boldsymbol{\varepsilon}_\zeta u(x - \zeta)$. In this notation, the Bernoulli-Anderson Hamiltonian is $H_{(\mathbb{Z}^d, \boldsymbol{\varepsilon}_{\mathbb{Z}^d})}$. If $\mathbf{Y}$ is a Poisson process on $\mathbb{R}^d$ with density 2ϱ, then $\mathbf{X} = \{\zeta \in \mathbf{Y};\, \boldsymbol{\varepsilon}_\zeta = 1\}$ is a Poisson process on $\mathbb{R}^d$ with density ϱ, and it follows that $H_{\mathbf{X}} = H_{(\mathbf{Y},\boldsymbol{\varepsilon}_{\mathbf{Y}})}$. Thus the Poisson Hamiltonian $H_{\mathbf{X}}$ can be rewritten as the Bernoulli-Poisson Hamiltonian $H_{(\mathbf{Y},\boldsymbol{\varepsilon}_{\mathbf{Y}})}$.

A very important difference between the Bernoulli-Anderson Hamiltonian and the Bernoulli-Poisson Hamiltonian is that, while for the former the impurities are placed on the fixed configuration $\mathbb{Z}^d$, for the latter the configuration of the impurities is random, being given by a Poisson process on $\mathbb{R}^d$. The randomness of the configuration must be taken care of in the multiscale analysis. Another difference is that the probability space for the Bernoulli-Anderson Hamiltonian is defined by a countable number of independent discrete (Bernoulli) random variables, but the probability space of a Poisson process is not so simple, leading to measurability questions absent in the case of the Bernoulli-Anderson Hamiltonian. The latter are of particular importance in this work as the Bourgain-Kenig multiscale analysis requires some detailed knowledge about the location of the impurities, as well as information on "free sites", and relies on conditional probabilities.

In order to control and keep track of the random location of the impurities, and also handle the measurability questions that appear for the Poisson process, we perform a finite volume reduction in each scale as part of the multiscale analysis, which estimates the probabilities of *good* boxes. We exploit properties of Poisson processes to construct, inside a box Λ_L, a scale dependent class of Λ_L-*acceptable* configurations of high probability for the Poisson process $\mathbf{Y}$ (Definition 3.4 and Lemma 3.5). We introduce an equivalence relation for Λ_L-acceptable configurations and, showing that we can move an impurity a little bit without spoiling the goodness of boxes (Lemma 3.3), we conclude that *goodness* of boxes is a property of equivalence classes of acceptable configurations (Lemma 3.6). Basic configurations and events in a given box are introduced in terms of these equivalence classes of acceptable configurations, and the multiscale analysis is performed for basic events. Thus we will have a new step in the multiscale analysis: basic configurations and events in a given box will have to be rewritten in terms of basic configurations and events in a bigger box (Lemma 3.13). The Wegner estimate at scale L is proved in Lemma 5.10 using [BoK, Lemma 5.1′].

Theorems 1.1 and 1.2 were announced in [GHK1]. Random Schrödinger operators with an attractive Poisson random potential, i.e., $H_{\mathbf{X}} = -\Delta - V_{\mathbf{X}}$ with $V_{\mathbf{X}}$ a Poisson random potential as in this paper, so $\sigma(H_{\mathbf{X}}) = \mathbb{R}$ with probability one, are studied in [GHK2], where we modify the methods of this paper to prove localization at low energies.

This paper is organized as follows. In Section 2 we describe the construction of a Poisson process $\mathbf{X}$ from a marked Poisson process $(\mathbf{Y}, \boldsymbol{\varepsilon}_{\mathbf{Y}})$, and review some useful deviation estimates for Poisson random variables. Section 3 is devoted to finite volume considerations and the control of Poisson configurations: We introduce finite volume operators, perform the finite volume reduction, study the effect of changing scales, and introduce localizing events. In Section 4 we prove *a priori* finite volume estimates that give the starting hypothesis for the multiscale analysis. Section 5 contains the multiscale analysis for Poisson Hamiltonians. Finally, the proofs of Theorems 1.1 and 1.2 are completed in Section 6.

2. Preliminaries

2.1. **Marked Poisson process.** We may assume that a Poisson process $\mathbf{X}$ on $\mathbb{R}^d$ with density ϱ is constructed from a marked Poisson process as follows: Consider a Poisson process $\mathbf{Y}$ on $\mathbb{R}^d$ with density 2ϱ, and to each $\zeta \in Y$ associate a Bernoulli random variable $\boldsymbol{\varepsilon}_\zeta$, either 0 or 1 with equal probability, with $\boldsymbol{\varepsilon}_{\mathbf{Y}} = \{\boldsymbol{\varepsilon}_\zeta\}_{\zeta \in \mathbf{Y}}$

independent random variables. Then $(\mathbf{Y}, \boldsymbol{\varepsilon}_{\mathbf{Y}})$ is a Poisson process with density 2ρ on the product space $\mathbb{R}^d \times \{0,1\}$, the *marked Poisson process*; its underlying probability space will still be denoted by $(\Omega, \mathbb{P})$. (We use the notation $(Y, \varepsilon_Y) := \{(\zeta, \varepsilon_\zeta);\, \zeta \in Y\} \in \mathcal{P}_0(\mathbb{R}^d \times \{0,1\})$. A Poisson process on $\mathbb{R}^d \times \{0,1\}$ with density $\mu > 0$ is a map $\tilde{\mathbf{Z}}$ from a probability space to $\mathcal{P}_0(\mathbb{R}^d \times \{0,1\})$, such that for each Borel set $\tilde{A} \subset \mathbb{R}^d \times \{0,1\}$ with $|\tilde{A}| := \frac{1}{2}(|\{x \in \mathbb{R}^d;\, (x,0) \in \tilde{A}\}| + |\{x \in \mathbb{R}^d;\, (x,1) \in \tilde{A}\}|) < \infty$, the random variable $N_{\tilde{\mathbf{Z}}}(\tilde{A})$ has Poisson distribution with mean $\mu|\tilde{A}|$, and the random variables $\{N_{\tilde{\mathbf{Z}}}(\tilde{A}_j)\}_{j=1}^n$ are independent for disjoint Borel subsets $\{\tilde{A}_j\}_{j=1}^n$. Define maps $\mathcal{X}, \mathcal{X}' \colon \mathcal{P}_0(\mathbb{R}^d \times \{0,1\}) \to \mathcal{P}_0(\mathbb{R}^d)$ by

$$\mathcal{X}(\tilde{Z}) := \{\zeta \in \mathbb{R}^d;\, (\zeta,1) \in \tilde{Z}\}, \quad \mathcal{X}'(\tilde{Z}) := \{\zeta \in \mathbb{R}^d;\, (\zeta,0) \in \tilde{Z}\}, \tag{2.1}$$

for all $\tilde{Z} \in \mathcal{P}_0(\mathbb{R}^d \times \{0,1\})$. Then the maps $\mathbf{X}, \mathbf{X}' \colon \Omega \to \mathcal{P}_0(\mathbb{R}^d)$, given by

$$\mathbf{X} := \mathcal{X}(\mathbf{Y}, \boldsymbol{\varepsilon}_{\mathbf{Y}}), \quad \mathbf{X}' := \mathcal{X}'(\mathbf{Y}, \boldsymbol{\varepsilon}_{\mathbf{Y}}), \tag{2.2}$$

i.e., $\mathbf{X}(\omega) = \mathcal{X}(\mathbf{Y}(\omega), \boldsymbol{\varepsilon}_{\mathbf{Y}(\omega)}(\omega))$, $\mathbf{X}'(\omega) = \mathcal{X}'(\mathbf{Y}(\omega), \boldsymbol{\varepsilon}_{\mathbf{Y}(\omega)}(\omega))$, are Poisson processes on $\mathbb{R}^d$ with density ϱ. (See [K, Section 5.2], [R, Example 2.4.2].) In particular, note that

$$N_{\mathbf{X}}(A) + N_{\mathbf{X}'}(A) = N_{\mathbf{Y}}(A) \quad \text{for all Borel sets } A \subset \mathbb{R}^d. \tag{2.3}$$

If $\mathbf{X}$ is a Poisson process on $\mathbb{R}^d$ with density ϱ, then $\mathbf{X}_A$ is a Poisson process on A with density ϱ for each Borel set $A \subset \mathbb{R}^d$, with $\{\mathbf{X}_{A_j}\}_{j=1}^n$ being independent Poisson processes for disjoint Borel subsets $\{A_j\}_{j=1}^n$. Similar considerations apply to $\mathbf{X}'$ and to the marked Poisson process $(\mathbf{Y}, \boldsymbol{\varepsilon}_{\mathbf{Y}})$, with $\mathbf{X}_A, \mathbf{X}'_A, \mathbf{Y}_A, \boldsymbol{\varepsilon}_{\mathbf{Y}_A}$ satisfying (2.2).

2.2. **Poisson random variables.** For a Poisson random variable N with mean μ we have (e.g., [K, Eq. (1.12)])

$$\mathbb{P}\{N \geq k\} = \int_0^\mu \mathrm{d}\lambda\, \frac{\lambda^{k-1}}{(k-1)!} \mathrm{e}^{-\lambda}, \quad \text{for } k = 1, 2, \ldots, \tag{2.4}$$

and hence also

$$\mathbb{P}\{N < k\} = \int_\mu^\infty \mathrm{d}\lambda\, \frac{\lambda^{k-1}}{(k-1)!} \mathrm{e}^{-\lambda}, \quad \text{for } k = 1, 2, \ldots. \tag{2.5}$$

From (2.4) we get useful upper and lower bounds:

$$\frac{\mu^k}{k!} \mathrm{e}^{-\mu} < \mathbb{P}\{N \geq k\} < \frac{\mu^k}{k!}, \quad \text{for } k = 1, 2, \ldots. \tag{2.6}$$

When $k > \mathrm{e}\mu > 1$, we can use a lower bound from Stirling's formula [Ro] to get

$$\mathbb{P}\{N \geq k\} < \frac{1}{\sqrt{2\pi k}} \left(\frac{\mathrm{e}\mu}{k}\right)^k. \tag{2.7}$$

In particular, if $\mathrm{e}\mu > 1$ and $a > \mathrm{e}^2$ we get the large deviation estimate

$$\mathbb{P}\{N \geq a\mu\} < \mathrm{e}^{-a\mu}. \tag{2.8}$$

From (2.5) we get

$$\mathbb{P}\{N < k\} < C_k \mathrm{e}^{-\frac{\mu}{2}}, \quad \text{with} \quad C_k = \int_0^\infty \mathrm{d}\lambda\, \frac{\lambda^{k-1}}{(k-1)!} \mathrm{e}^{-\frac{\lambda}{2}} \quad \text{for } k = 1, 2, \ldots. \tag{2.9}$$

3. Finite volume and Poisson configurations

From now on $H_{\mathbf{X}}$ will always denote a Poisson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ with density $\varrho > 0$, as in (1.1)-(1.3). We recall that $(\Omega, \mathbb{P})$ is the underlying probability space on which the Poisson processes $\mathbf{X}$ and $\mathbf{X}'$, with density ϱ, and $\mathbf{Y}$, with density 2ϱ, are defined, as well as the Bernoulli random variables $\boldsymbol{\varepsilon}_{\mathbf{Y}}$, and we have (2.2). All events will be defined with respect to this probability space. We will use the notation $\sqcup$ for disjoint unions: $C = A \sqcup B$ means $C = A \cup B$ with $A \cap B = \emptyset$.

Given two disjoint configurations $X, Y \in \mathcal{P}_0(\mathbb{R}^d)$ and $t_Y = \{t_\zeta\}_{\zeta \in Y} \in [0,1]^Y$, we set

$$H_{X,(Y,t_Y)} := -\Delta + V_{X,(Y,t_Y)}, \text{ where } V_{X,(Y,t_Y)}(x) := V_X(x) + \sum_{\zeta \in Y} t_\zeta u(x-\zeta). \tag{3.1}$$

In particular, given $\varepsilon_Y \in \{0,1\}^Y$ we have, recalling (2.1), that

$$H_{X,(Y,\varepsilon_Y)} = H_{X \sqcup \mathcal{X}(Y,\varepsilon_Y)}. \tag{3.2}$$

We also write $H_{(Y,t_Y)} := H_{\emptyset,(Y,t_Y)}$ and

$$H_\omega := H_{\mathbf{X}(\omega)} = H_{(\mathbf{Y}(\omega), \boldsymbol{\varepsilon}_{\mathbf{Y}(\omega)}(\omega))}. \tag{3.3}$$

3.1. **Finite volume operators.** Finite volume operators are defined as follows: Given a box $\Lambda = \Lambda_L(x)$ in $\mathbb{R}^d$ and a configuration $X \in \mathcal{P}_0(\mathbb{R}^d)$, we set

$$H_{X,\Lambda} := -\Delta_\Lambda + V_{X,\Lambda} \quad \text{on} \quad \mathrm{L}^2(\Lambda), \tag{3.4}$$

where Δ_Λ is the Laplacian on Λ with Dirichlet boundary condition, and

$$V_{X,\Lambda} := \chi_\Lambda V_{X_\Lambda} \quad \text{with } V_{X_\Lambda} \text{ as in (1.2).} \tag{3.5}$$

The finite volume resolvent is $R_{X,\Lambda}(z) := (H_{X,\Lambda} - z)^{-1}$.

We have $\Delta_\Lambda = \nabla_\Lambda \cdot \nabla_\Lambda$, where ∇_Λ is the gradient with Dirichlet boundary condition. We sometimes identify $\mathrm{L}^2(\Lambda)$ with $\chi_\Lambda \mathrm{L}^2(\mathbb{R}^d)$ and, when necessary, will use subscripts Λ and $\mathbb{R}^d$ to distinguish between the norms and inner products of $\mathrm{L}^2(\Lambda)$ and $\mathrm{L}^2(\mathbb{R}^d)$. Note that in general we do not have $V_{X,\Lambda} = \chi_\Lambda V_{X,\Lambda'}$ for $\Lambda \subset \Lambda'$, where Λ' may be a finite box or $\mathbb{R}^d$. But we always have

$$\chi_{\widehat{\Lambda}} V_{X,\Lambda} = \chi_{\widehat{\Lambda}} V_{X,\Lambda'}, \tag{3.6}$$

where

$$\widehat{\Lambda} = \widehat{\Lambda}_L(x) := \Lambda_{L-\delta_+}(x) \quad \text{with } \delta_+ \text{ as in (1.3),} \tag{3.7}$$

which suffices for the multiscale analysis.

The multiscale analysis estimates probabilities of desired properties of finite volume resolvents at energies $E \in \mathbb{R}$. (By $L^{p\pm}$ we mean $L^{p\pm\delta}$ for some small $\delta > 0$, fixed independently of the scale.)

Definition 3.1. *Consider an energy $E \in \mathbb{R}$, a rate of decay $m > 0$, and a configuration $X \in \mathcal{P}_0(\mathbb{R}^d)$. A box Λ_L is said to be (X, E, m)-good if*

$$\|R_{X,\Lambda_L}(E)\| \le \mathrm{e}^{L^{1-}} \tag{3.8}$$

and

$$\|\chi_x R_{X,\Lambda_L}(E) \chi_y\| \le \mathrm{e}^{-m|x-y|}, \quad \textit{for all } x, y \in \Lambda_L \textit{ with } |x-y| \ge \tfrac{L}{10}. \tag{3.9}$$

The box Λ_L is (ω, E, m)-good if it is $(\mathbf{X}(\omega), E, m)$-good.

Note that [BoK, Lemmas 2.14] requires condition (3.9) as stated above for its proof.

But *goodness* of boxes does not suffice for the induction step in the multiscale analysis given in [Bo, BoK], which also needs an adequate supply of *free sites* to obtain a Wegner estimate at each scale. Given two disjoint configurations $X, Y \in \mathcal{P}_0(\mathbb{R}^d)$ and $t_Y = \{t_\zeta\}_{\zeta \in Y} \in [0,1]^Y$, we recall (3.1) and define the corresponding finite volume operators $H_{X,(Y,t_Y),\Lambda}$ as in (3.4) and (3.5) using X_Λ, Y_Λ and t_{Y_Λ}, i.e.,

$$H_{X,(Y,t_Y),\Lambda} := -\Delta_\Lambda + V_{X,(Y,t_Y),\Lambda}, \quad \text{where} \quad V_{X,(Y,t_Y),\Lambda} := \chi_\Lambda V_{X_\Lambda,(Y_\Lambda,t_{Y_\Lambda})}, \tag{3.10}$$

with $R_{X,(Y,t_Y),\Lambda}(z)$ being the corresponding finite volume resolvent.

Definition 3.2. *Consider an energy $E \in \mathbb{R}$, a rate of decay $m > 0$, and two configurations $X, Y \in \mathcal{P}_0(\mathbb{R}^d)$. A box Λ_L is said to be (X, Y, E, m)-good if $X \cap Y = \emptyset$ and we have* (3.8) *and* (3.9) *with $R_{X,(Y,t_Y),\Lambda_L}(E)$ for all $t_Y \in [0,1]^Y$. In this case Y consists of (X, E)-free sites for the box Λ_L.* (In particular, the box Λ_L is $(X \sqcup \mathcal{X}(Y, \varepsilon_Y), E, m)$-good for all $\varepsilon_Y \in \{0,1\}^Y$.)

3.2. **Finite volume reduction of Poisson configurations.** The multiscale analysis will require some detailed knowledge about the location of the impurities, that is, about the Poisson process configuration, as well as information on "free sites". To do so and also handle the measurability questions that appear for the Poisson process we will perform a finite volume reduction as part of the multiscale analysis. The key is that we can move a Poisson point a little bit without spoiling the goodness of boxes, using the following lemma.

Lemma 3.3. *Let Λ be a box in $\mathbb{R}^d$, $0 \le W \in \mathrm{L}^1_{\mathrm{loc}}(\Lambda)$, $0 \le w \in \mathrm{L}^\infty(\Lambda)$ with compact support. Given $\zeta \in \Lambda^{(w)} = \{\zeta \in \Lambda;\ \operatorname{supp} w(\cdot - \zeta) \subset \Lambda\}$, let $H_\zeta = -\Delta_\Lambda + W + w(\cdot - \zeta)$ on $\mathrm{L}^2(\Lambda)$, with $R_\zeta(z) = (H_\zeta - z)^{-1}$ its resolvent.*

(i) *Suppose that for some $\zeta \in \Lambda^{(w)}$, $E \ge 0$, and $\gamma \ge 1$ we have $\|R_\zeta(E)\| \le \gamma$, and let*

$$0 < \eta \le \min\left\{ \left(4\sqrt{1+E}\, \|w\|_\infty\, \gamma\right)^{-2}, \tfrac{1}{4} \right\}. \tag{3.11}$$

Then for all $\zeta' \in \Lambda^{(w)}$ with $|\zeta' - \zeta| \le \eta$ we have

$$\|R_{\zeta'}(E)\| \le \mathrm{e}^{\sqrt{\eta}}\gamma \tag{3.12}$$

and

$$\|\chi_x R_{\zeta'}(E)\chi_y\| \le \|\chi_x R_\zeta(E)\chi_y\| + \sqrt{\eta}\,\gamma, \quad \textit{for all } x, y \in \Lambda. \tag{3.13}$$

(ii) *Suppose that for some $\zeta \in \Lambda^{(w)}$, $E \ge 0$, and $\beta \ge 2$ we have $\operatorname{dist}(E, \sigma(H_\zeta)) \le \beta^{-1}$, i.e., $\|R_\zeta(E)\| \ge \beta$, and let η be as in* (3.11) *with β substituted for γ. Then for all $\zeta' \in \Lambda^{(w)}$ with $|\zeta' - \zeta| \le \eta$ we have*

$$\|R_{\zeta'}(E)\| \ge \mathrm{e}^{-\sqrt{\eta}}\beta, \quad \textit{i.e.,} \quad \operatorname{dist}(E, \sigma(H_{\zeta'})) \le \mathrm{e}^{\sqrt{\eta}}\beta^{-1}. \tag{3.14}$$

Proof. We set $R = R_\zeta(E)$, $R' = R_{\zeta'}(E)$, $u = w(\cdot - \zeta)$, $u' = w(\cdot - \zeta')$, and $\xi = \zeta' - \zeta$ with $|\xi| \le \eta$. We let $U(a)$ denote translation by a in $L^2(\mathbb{R}^d)$: $(U(a)\varphi)(x) = \varphi(x - a)$, and pick $\phi \in C_c^\infty(\Lambda)$ such that $0 \le \phi \le 1$ and $\phi \equiv 1$ in some open subset of Λ

which contains the supports of u and u'. It follows from the resolvent identity that

$$
\begin{aligned}
\|R'\|_\Lambda - \|R\|_\Lambda &\le \|R'(u'-u)R\|_\Lambda = \|\chi_\Lambda R'\phi(u'-u)\phi R\chi_\Lambda\|_{\mathbb{R}^d} \\
&= \|\chi_\Lambda R'\phi(U(\xi)uU(\xi)^* - u)\phi R\chi_\Lambda\|_{\mathbb{R}^d} \qquad (3.15)\\
&\le \|\chi_\Lambda R'\phi(U(\xi)-1)uU(\xi)^*\phi R\chi_\Lambda\|_{\mathbb{R}^d} + \|\chi_\Lambda R'\phi u(U(\xi)^*-1)\phi R\chi_\Lambda\|_{\mathbb{R}^d} \\
&\le \eta\left(\|u\nabla\phi R'\chi_\Lambda\|_{\mathbb{R}^d}\|\phi R\chi_\Lambda\|_{\mathbb{R}^d} + \|\phi R'\chi_\Lambda\|_{\mathbb{R}^d}\|u\nabla\phi R\chi_\Lambda\|_{\mathbb{R}^d}\right) \\
&= \eta\left(\|u\nabla_\Lambda\phi R'\|_\Lambda\|\phi R\|_\Lambda + \|\phi R'\|_\Lambda\|u\nabla_\Lambda\phi R\|_\Lambda\right) \\
&\le \eta\|u\|_\infty\left(\|\nabla_\Lambda R'\|_\Lambda\|R\|_\Lambda + \|R'\|_\Lambda\|\nabla_\Lambda R\|_\Lambda\right) \\
&\le 2\sqrt{1+E}\,\|u\|_\infty\,\eta\max\{\|R\|_\Lambda,1\}\max\{\|R'\|_\Lambda,1\},
\end{aligned}
$$

where we used

$$
\left\|\nabla_\Lambda R^\sharp\right\|_\Lambda^2 \le \left\|R^\sharp\right\|_\Lambda + E\left\|R^\sharp\right\|_\Lambda^2 \le (1+E)\max\{\left\|R^\sharp\right\|_\Lambda^2,1\} \text{ for } R^\sharp = R, R'. \qquad (3.16)
$$

To prove part (i), if $\|R\|_\Lambda \le \gamma$ with $\gamma \ge 1$, it follows from (3.15) and (3.11) that

$$
\|R'\|_\Lambda - \|R\|_\Lambda \le \|R'(u'-u)R\|_\Lambda \le \tfrac{1}{2}\sqrt{\eta}\max\{\|R'\|_\Lambda,1\}. \qquad (3.17)
$$

To prove (3.12), we may assume that $\|R'\|_\Lambda \ge 1$, since otherwise the result is trivial. The estimate (3.12) now follows immediately from (3.17) and (3.11). Using the resolvent identity, (3.17), (3.12), and $\frac{1}{2}\mathrm{e}^{\frac{1}{2}} < 1$ we get (3.13).

Part (ii) follows from part (i) as follows. Let $\beta \ge 2$ and suppose (3.14) does not hold, i.e., $\|R'\|_\Lambda < \mathrm{e}^{-\sqrt{\eta}}\beta$. Since $\mathrm{e}^{-\sqrt{\eta}}\beta \ge \mathrm{e}^{-\frac{1}{2}}2 > 1$, we may apply (3.12) to get a contradiction to $\|R\|_\Lambda \ge \beta$, namely $\|R\|_\Lambda < \mathrm{e}^{\sqrt{\eta}}\left(\mathrm{e}^{-\sqrt{\eta}}\beta\right) = \beta$. □

Lemma 3.3 lets us move one Poisson point a little bit, namely by η, and maintain good bounds on the resolvent. Since we will want to preserve the "goodness" of the box $\Lambda = \Lambda_L$, we will use Lemma 3.3 with $\gamma = \mathrm{e}^{L^{1-}}$ (as in (3.8)), and take $\eta \ll \mathrm{e}^{-L}$. To fix ideas we set $\eta = \mathrm{e}^{-L^{10^6 d}}$. To move *all* Poisson points in Λ_L we will need to control the number of Poisson points in the box. Moreover, we will have to know the location of these Poisson points with good precision. That this can be done at very little cost in probability is the subject of the next lemma.

Definition 3.4. *Let $\eta_L := \mathrm{e}^{-L^{10^6 d}}$ for $L > 0$. Given a box $\Lambda = \Lambda_L(x)$, set*

$$
\mathbb{J}_\Lambda := \{j \in x + \eta_L\mathbb{Z}^d;\ \Lambda_{\eta_L}(j) \subset \Lambda\}. \qquad (3.18)
$$

A configuration $X \in \mathcal{P}_0(\mathbb{R}^d)$ is said to be Λ-acceptable if

$$
N_X(\Lambda) < 16\varrho L^d, \qquad (3.19)
$$

$$
N_X(\Lambda_{\eta_L}(j)) \le 1, \quad \textit{for all} \quad j \in \mathbb{J}_\Lambda, \qquad (3.20)
$$

and

$$
N_X(\Lambda \setminus \sqcup_{j\in\mathbb{J}_\Lambda}\Lambda_{\eta_L(1-\eta_L)}(j)) = 0; \qquad (3.21)
$$

it is Λ-acceptable$'$ if it satisfies (3.19),(3.20), and the less restrictive

$$
N_X(\Lambda \setminus \sqcup_{j\in\mathbb{J}_\Lambda}\Lambda_{\eta_L}(j)) = 0. \qquad (3.22)
$$

We set

$$
\mathcal{Q}_\Lambda^{(0)} := \{X \in \mathcal{P}_0(\mathbb{R}^d);\ X \textit{ is } \Lambda\textit{-acceptable}\}, \qquad (3.23)
$$

$$
\mathcal{Q}_\Lambda^{(0\prime)} := \{X \in \mathcal{P}_0(\mathbb{R}^d);\ X \textit{ is } \Lambda\textit{-acceptable}'\}, \qquad (3.24)
$$

and consider the event (recall that $\mathbf{Y}$ *is the Poisson process with density* 2ϱ*)*

$$\Omega_\Lambda^{(0)} := \{\mathbf{Y} \in \mathcal{Q}_\Lambda^{(0)}\}. \tag{3.25}$$

Note that $\Omega_\Lambda^{(0)} \subset \{\mathbf{X} \in \mathcal{Q}_\Lambda^{(0)}\}$ in view of (2.3) and $\mathcal{Q}_\Lambda^{(0)} \subset \mathcal{Q}_\Lambda^{(0\prime)}$. We require condition (3.21) for acceptable configurations to avoid ambiguities when changing scales (cf. Lemma 3.13), but we will then need Lemma 3.6 for acceptable$'$ configurations.

We now impose a condition on ϱ and L that will be always satisfied when we do the multiscale analysis:

$$L^{-(0+)} \leq \varrho \leq \mathrm{e}^{L^d}. \tag{3.26}$$

From now on we assume (3.26).

Lemma 3.5. *There exists a scale* $\overline{L} = \overline{L}(d) < \infty$*, such that if* $L \geq \overline{L}$ *we have*

$$\mathbb{P}\{\Omega_{\Lambda_L}^{(0)}\} \geq 1 - \mathrm{e}^{-L^{d-}}. \tag{3.27}$$

Proof. Using (2.8) and (2.6) we get

$$\mathbb{P}\{\Omega_{\Lambda_L}^{(0)}\} \geq 1 - \mathrm{e}^{-16\varrho L^d} - 4d\varrho(L^{d-1} + L^d)\eta_L - 2\varrho^2 L^d \eta_L^d, \tag{3.28}$$

and hence (3.27) follows for large L using (3.26). □

Lemma 3.5 tells us that inside the box Λ, outside an event of negligible probability in the multiscale analysis, we only need to consider Λ-acceptable configurations of the Poisson process $\mathbf{Y}$.

Given a box $\Lambda = \Lambda_L(x)$, we define an *equivalence relation for configurations* by

$$X \overset{\Lambda}{\sim} Z \Longleftrightarrow N_X(\Lambda_{\eta_L}(j)) = N_Z(\Lambda_{\eta_L}(j)) \quad \text{for all} \quad j \in \mathbb{J}_\Lambda. \tag{3.29}$$

This induces an equivalence relation in both $\mathcal{Q}_\Lambda^{(0\prime)}$ and $\mathcal{Q}_\Lambda^{(0)}$; the equivalence class of X in $\mathcal{Q}_\Lambda^{(0\prime)}$ will be denoted by $[X]_\Lambda'$. If $X \in \mathcal{Q}_\Lambda^{(0)}$, then $[X]_\Lambda = [X]_\Lambda' \cap \mathcal{Q}_\Lambda^{(0)}$ is its equivalence class in $\mathcal{Q}_\Lambda^{(0)}$. Note that $[X]_\Lambda' = [X_\Lambda]_\Lambda'$. We also write

$$[A]_\Lambda := \bigcup_{X \in A} [X]_\Lambda \quad \text{for subsets} \quad A \subset \mathcal{Q}_\Lambda^{(0)}. \tag{3.30}$$

The following lemma is an immediate consequence of Lemma 3.3(i); it tells us that "goodness" of boxes is a property of equivalence classes of acceptable$'$ configurations: changing configurations inside an equivalence class takes good boxes into just-as-good (jgood) boxes.

Lemma 3.6. *Fix* $E_0 > 0$ *and consider an energy* $E \in [0, E_0]$*. Suppose the box* $\Lambda = \Lambda_L$ *(with L large) is* (X, E, m)*-good for some* $X \in \mathcal{Q}_{\Lambda_L}^{(0\prime)}$*. Then for all* $Z \in [X]_\Lambda'$ *the box* Λ *is* (Z, E, m)*-jgood (for just-as-good), that is,*

$$\|R_{Z,\Lambda}(E)\| \leq \mathrm{e}^{L^{1-} + \eta_L^{\frac{1}{4}}} \sim \mathrm{e}^{L^{1-}} \tag{3.31}$$

and

$$\|\chi_x R_{Z,\Lambda}(E) \chi_y\| \leq \mathrm{e}^{-m|x-y|} + \eta_L^{\frac{1}{4}} \sim \mathrm{e}^{-m|x-y|}, \quad \text{for } x, y \in \Lambda \text{ with } |x-y| \geq \tfrac{L}{10}. \tag{3.32}$$

Moreover, if $X, Z, X \sqcup Z \in \mathcal{Q}_\Lambda^{(0\prime)}$ *and the box* Λ *is* (X, Z, E, m)*-good, then for all* $X_1 \in [X]_\Lambda'$ *and* $Z_1 \in [Z]_\Lambda'$ *we have* $X_1 \sqcup Z_1 \in [X \sqcup Z]_\Lambda'$*, and the box* Λ *is* (X_1, Z_1, E, m)*-jgood as in* (3.31) *and* (3.32).

Proof. Lemma 3.3(i) gives

$$\|R_{X',\Lambda}(E)\| \leq \mathrm{e}^{L^{1-}+16\varrho L^d\sqrt{\eta_L}}, \tag{3.33}$$

and, for all $x, y \in \Lambda$ with $|x-y| \geq \frac{L}{10}$,

$$\|\chi_x R_{X',\Lambda}(E)\chi_y\| \leq \mathrm{e}^{-m|x-y|} + 16\varrho L^d\sqrt{\eta_L}\,\mathrm{e}^{L^{1-}+16\varrho L^d\sqrt{\eta_L}}. \tag{3.34}$$

Using (3.26), we get (3.31) and (3.32) for large L.

The remaining statement is immediate. □

Remark 3.7. *Proceeding as in Lemma 3.6, we find that changing configurations inside an equivalence class takes jgood boxes into what we may call just-as-just-as-good (jjgood) boxes, and so on. Since we will only carry this procedure a bounded number of times, the bound independent of the scale, we will simply call them all jgood boxes.*

Similarly, we get the following consequence of Lemma 3.3(ii).

Lemma 3.8. *Fix $E_0 > 0$ and consider an energy $E \in [0, E_0]$ and a box $\Lambda = \Lambda_L$ (with L large). Suppose $\operatorname{dist}(E, \sigma(H_{X,\Lambda})) \leq \tau_L$ for some $X \in \mathcal{Q}_{\Lambda_L}^{(0\prime)}$, where $\sqrt{\eta_L} \ll \tau_L < \frac{1}{2}$. Then*

$$\operatorname{dist}(E, \sigma(H_{Y,\Lambda})) \leq \mathrm{e}^{\eta_L^{\frac{1}{4}}}\,\tau_L, \quad \textit{for all} \quad Y \in [X]'_\Lambda. \tag{3.35}$$

In view of (3.19)-(3.20) we have

$$\mathcal{Q}_\Lambda^{(0)}/\overset{\Lambda}{\sim} = \{[J]_\Lambda;\ J \in \mathcal{J}_\Lambda\}, \quad \text{where} \quad \mathcal{J}_\Lambda := \{J \subset \mathbb{J}_\Lambda;\ \#J < 16\varrho L^d\}, \tag{3.36}$$

and we can write $\mathcal{Q}_\Lambda^{(0)}$ and $\Omega_\Lambda^{(0)}$ as

$$\mathcal{Q}_\Lambda^{(0)} = \bigsqcup_{J\in\mathcal{J}_\Lambda} [J]_\Lambda \quad \text{and} \quad \Omega_\Lambda^{(0)} = \bigsqcup_{J\in\mathcal{J}_\Lambda} \{\mathbf{Y} \in [J]_\Lambda\}. \tag{3.37}$$

3.3. **Basic events.** The multiscale analysis will require "free sites" and sub-events of $\{\mathbf{Y} \in [J]_\Lambda\}$.

Definition 3.9. *Given $\Lambda = \Lambda_L(x)$, a Λ-bconfset (basic configuration set) is a subset of $\mathcal{Q}_\Lambda^{(0)}$ of the form*

$$C_{\Lambda,B,S} := \bigsqcup_{\varepsilon_S\in\{0,1\}^S} [B \cup \mathcal{X}(S,\varepsilon_S)]_\Lambda = \bigsqcup_{S'\subset S} [B \cup S']_\Lambda, \tag{3.38}$$

where we always implicitly assume $B \sqcup S \in \mathcal{J}_\Lambda$. $C_{\Lambda,B,S}$ is a Λ-dense bconfset if S satisfies the density condition (cf. (3.7))

$$\#(S \cap \widehat{\Lambda}_{L^{1-}}) \geq L^{d-}, \quad \textit{for all boxes} \quad \Lambda_{L^{1-}} \subset \Lambda_L. \tag{3.39}$$

We also set

$$C_{\Lambda,B} := C_{\Lambda,B,\emptyset} = [B]_\Lambda. \tag{3.40}$$

Definition 3.10. *Given $\Lambda = \Lambda_L(x)$, a Λ-bevent (basic event) is a subset of $\Omega_\Lambda^{(0)}$ of the form*

$$\mathcal{C}_{\Lambda,B,B',S} := \{\mathbf{Y} \in [B \sqcup B' \sqcup S]_\Lambda\} \cap \{\mathbf{X} \in C_{\Lambda,B,S}\} \cap \{\mathbf{X}' \in C_{\Lambda,B',S}\}, \tag{3.41}$$

where we always implicitly assume $B \sqcup B' \sqcup S \in \mathcal{J}_\Lambda$. In other words, the Λ-bevent $\mathcal{C}_{\Lambda,B,B',S}$ consists of all $\omega \in \Omega_\Lambda^{(0)}$ satisfying

$$\begin{array}{lll} N_{\mathbf{X}(\omega)}(\Lambda_{\eta_L}(j)) = 1 & \text{if} & j \in B, \\ N_{\mathbf{X}'(\omega)}(\Lambda_{\eta_L}(j)) = 1 & \text{if} & j \in B', \\ N_{\mathbf{Y}(\omega)}(\Lambda_{\eta_L}(j)) = 1 & \text{if} & j \in S, \\ N_{\mathbf{Y}(\omega)}(\Lambda_{\eta_L}(j)) = 0 & \text{if} & j \in \mathbb{J}_\Lambda \backslash (B \sqcup B' \sqcup S). \end{array} \tag{3.42}$$

$\mathcal{C}_{\Lambda,B,B',S}$ is a Λ-dense bevent if S satisfies the density condition (3.39)*. In addition, we set*

$$\mathcal{C}_{\Lambda,B,B'} := \mathcal{C}_{\Lambda,B,B',\emptyset} = \{\mathbf{X} \in C_{\Lambda,B}\} \cap \{\mathbf{X}' \in C_{\Lambda,B'}\}. \tag{3.43}$$

The number of possible bconfsets and bevents in a given box is always finite. We always have

$$\mathcal{C}_{\Lambda,B,B',S} \subset \{\mathbf{X} \in C_{\Lambda,B,S}\} \cap \Omega_\Lambda^{(0)}, \tag{3.44}$$

$$\mathcal{C}_{\Lambda,B,B',S} \subset \mathcal{C}_{\Lambda,\emptyset,\emptyset,B\sqcup B'\sqcup S} = \{\mathbf{Y} \in [B \sqcup B' \sqcup S]_\Lambda\}. \tag{3.45}$$

Note also that it follows from (3.25), (3.36) and (3.43) that

$$\Omega_\Lambda^{(0)} = \bigsqcup_{\{(B,B');\, B\sqcup B' \in \mathcal{J}_\Lambda\}} \mathcal{C}_{\Lambda,B,B'} \tag{3.46}$$

Moreover, for each $S_1 \subset S$ we have

$$C_{\Lambda,B,S} = \bigsqcup_{S_2 \subset S_1} C_{\Lambda,B\sqcup S_2,S\backslash S_1}, \tag{3.47}$$

$$\mathcal{C}_{\Lambda,B,B',S} = \bigsqcup_{S_2 \subset S_1} \mathcal{C}_{\Lambda,B\sqcup S_2,B'\sqcup(S_1\backslash S_2),S\backslash S_1}. \tag{3.48}$$

In view of Lemma 3.6, we make the following definition.

Definition 3.11. *Consider an energy $E \in \mathbb{R}$, $m > 0$, and a box $\Lambda = \Lambda_L(x)$. The Λ-bevent $\mathcal{C}_{\Lambda,B,B',S}$ and the Λ-bconfset $C_{\Lambda,B,S}$ are (Λ, E, m)-good if the box Λ is (B, S, E, m)-good.* (Note that Λ is then (ω, E, m)-jgood for every $\omega \in \mathcal{C}_{\Lambda,B,B',S}$.) *Those (Λ, E, m)-good bevents and bconfsets that are also Λ-dense will be called (Λ, E, m)-adapted.*

3.4. **Changing scales.** Since the finite volume reduction is scale dependent, it introduces new considerations in the multiscale analysis for Poisson Hamiltonians. Given $\Lambda_\ell \subset \Lambda$, the multiscale analysis will require us to redraw Λ_ℓ-bevents and bconfsets in terms of (Λ, Λ_ℓ)-bevents and bconfsets as follows.

Definition 3.12. *Given $\Lambda_\ell \subset \Lambda$, a configuration $J \in \mathcal{J}_\Lambda$ is called Λ_ℓ-compatible if*

$$J \cap \Lambda_\ell \in \mathcal{J}_\Lambda^{\Lambda_\ell} := \bigsqcup_{A \in \mathcal{J}_{\Lambda_\ell}} \mathbb{J}_\Lambda(A) \subset \mathcal{J}_\Lambda, \tag{3.49}$$

where

$$\mathbb{J}_\Lambda(A) := \{J \subset \mathbb{J}_\Lambda \cap \Lambda_\ell;\, J \in [A]_{\Lambda_\ell}\} \quad \text{for} \quad A \subset \mathbb{J}_{\Lambda_\ell}. \tag{3.50}$$

If $B \sqcup S$ is Λ_ℓ-compatible, the Λ-bconfset $C_{\Lambda,B,S}$ is also called Λ_ℓ-compatible, and we define the (Λ, Λ_ℓ)-bconfset

$$C_{\Lambda,B,S}^{\Lambda_\ell} := \{X \in \mathcal{P}_0(\mathbb{R}^d);\, X_{\Lambda_\ell} \in C_{\Lambda,B\cap\Lambda_\ell,S\cap\Lambda_\ell}\} \subset \mathcal{Q}_{\Lambda_\ell}^{(0\prime)}. \tag{3.51}$$

If $B \sqcup B' \sqcup S$ is Λ_ℓ-compatible, the Λ-bevent $\mathcal{C}_{\Lambda,B,B',S}$ is also called Λ_ℓ-compatible, and we define the (Λ, Λ_ℓ)-bevent

$$\mathcal{C}^{\Lambda_\ell}_{\Lambda,B,B',S} := \{\mathbf{Y}_{\Lambda_\ell} \in [(B \sqcup B' \sqcup S) \cap \Lambda_\ell]_\Lambda\} \cap \{\mathbf{X}_{\Lambda_\ell} \in C^{\Lambda_\ell}_{\Lambda,B,S}\} \cap \{\mathbf{X}'_{\Lambda_\ell} \in C^{\Lambda_\ell}_{\Lambda,B',S}\}. \tag{3.52}$$

Moreover, we say that a Λ_ℓ-compatible Λ-bconfset $C_{\Lambda,B,S}$ or a Λ-bevent $\mathcal{C}_{\Lambda,B,B',S}$ is (Λ, Λ_ℓ)-dense if $S \cap \Lambda_\ell$ satisfies the density condition (3.39) *in Λ_ℓ; $(\Lambda, \Lambda_\ell, E, m)$-jgood if the box Λ_ℓ is (B, S, E, m)-jgood; $(\Lambda, \Lambda_\ell, E, m)$-adapted if both (Λ, Λ_ℓ)-dense and $(\Lambda, \Lambda_\ell, E, m)$-jgood.* (Note that whenever we define a property of a Λ-bconfset or bevent on a subbox $\Lambda_\ell \subset \Lambda$ we will always implicitly assume Λ_ℓ-compatibility.)

Lemma 3.13. *Let $\Lambda_\ell \subset \Lambda$. Then for all Λ_ℓ-bconfsets $C_{\Lambda_\ell,B,S}$ and Λ_ℓ-bevents $\mathcal{C}_{\Lambda_\ell,B,B',S}$ we have*

$$C_{\Lambda_\ell,B,S} \cap \mathcal{Q}^{(0)}_\Lambda \subset \bigcup_{B_1 \in \mathbb{J}_\Lambda(B),\, S_1 \in \mathbb{J}_\Lambda(S)} C^{\Lambda_\ell}_{\Lambda,B_1,S_1}, \tag{3.53}$$

$$\mathcal{C}_{\Lambda_\ell,B,B',S} \cap \Omega^{(0)}_\Lambda \subset \bigsqcup_{B_1 \in \mathbb{J}_\Lambda(B), B'_1 \in \mathbb{J}_\Lambda(B'), S_1 \in \mathbb{J}_\Lambda(S)} \mathcal{C}^{\Lambda_\ell}_{\Lambda,B_1,B'_1,S_1}. \tag{3.54}$$

Moreover, if $C_{\Lambda_\ell,B,S}$ or $\mathcal{C}_{\Lambda_\ell,B,B',S}$ is Λ_ℓ-dense, or (Λ_ℓ, E, m)-jgood, or (Λ_ℓ, E, m)-adapted, then then each $C^{\Lambda_\ell}_{\Lambda,B_1,S_1}$ or $\mathcal{C}^{\Lambda_\ell}_{\Lambda,B_1,B'_1,S_1}$ is (Λ, Λ_ℓ)-dense, or $(\Lambda, \Lambda_\ell, E, m)$-jgood, or $(\Lambda, \Lambda_\ell, E, m)$-adapted.

Proof. If $C_{\Lambda_\ell,B,S}$ is a Λ_ℓ-bconfset, then $\{C^{\Lambda_\ell}_{\Lambda,B_1,S_1}\}_{B_1 \in \mathbb{J}_\Lambda(B),\, S_1 \in \mathbb{J}_\Lambda(S)}$ form a collection of (not necessarily disjoint) (Λ, Λ_ℓ)-bconfsets, and we have (3.53). The same argument yields (3.54), but now the (Λ, Λ_ℓ)-bevents are disjoint. (There are no ambiguities since $\eta_L \ll \sqrt{\eta_\ell}$ and we have condition (3.21) at both scales.) The rest follows, using also Lemma 3.6. □

3.5. **Localizing events.**

Definition 3.14. *Consider an energy $E \in \mathbb{R}$, a rate of decay $m > 0$, and a box Λ. We call Ω_Λ a (Λ, E, m)-localized event if there exist disjoint (Λ, E, m)-adapted bevents $\{\mathcal{C}_{\Lambda,B_i,B'_i,S_i}\}_{i=1,2,\ldots,I}$ such that*

$$\Omega_\Lambda = \bigsqcup_{i=1}^{I} \mathcal{C}_{\Lambda,B_i,B'_i,S_i}. \tag{3.55}$$

If Ω_Λ is a (Λ, E, m)-localized event, note that $\Omega_\Lambda \subset \Omega^{(0)}_\Lambda$ by its definition, and hence, recalling (3.48) and (3.43) , we can rewrite Ω_Λ in the form

$$\Omega_\Lambda = \bigsqcup_{j=1}^{J} \mathcal{C}_{\Lambda,A_j,A'_j}, \tag{3.56}$$

where the $\{\mathcal{C}_{\Lambda,A_j,A'_j}\}_{j=1,2,\ldots,J}$ are disjoint (Λ, E, m)-good bevents.

We will need (Λ, E, m)-localized events of scale appropriate probability.

Definition 3.15. *Fix $p > 0$. Given an energy $E \in \mathbb{R}$ and a rate of decay $m > 0$, a scale L is (E, m)-localizing if for some box $\Lambda = \Lambda_L$ (and hence for all) we have a (Λ, E, m)-localized event Ω_Λ such that*

$$\mathbb{P}\{\Omega_\Lambda\} > 1 - L^{-p}. \tag{3.57}$$

In Section 6 we will also need "just localizing" events and scales.

Definition 3.16. *Consider an energy $E \in \mathbb{R}$, a rate of decay $m > 0$, and a box Λ. We call Ω_Λ a (Λ, E, m)-jlocalized event if there exist disjoint (Λ, E, m)-good bevents $\{\mathcal{C}_{\Lambda, A_j, A'_j}\}_{j=1,2,\ldots,J}$ such that*

$$\Omega_\Lambda = \bigsqcup_{j=1}^{J} \mathcal{C}_{\Lambda, A_j, A'_j}. \tag{3.58}$$

A scale L is (E, m)-jlocalizing if for some box $\Lambda = \Lambda_L$ (and hence for all) we have a (Λ, E, m)-jlocalized event Ω_Λ such that

$$\mathbb{P}\{\Omega_\Lambda\} > 1 - L^{-p}. \tag{3.59}$$

An (E, m)-localizing scale L is (E, m)-jlocalizing in view of (3.56).

4. "A priori" finite volume estimates

Given an energy E, to start the multiscale analysis we will need, as in [Bo, BoK], an *a priori* estimate on the probability that a box Λ_L is good with an adequate supply of free sites, for some sufficiently large scale L. The multiscale analysis will then show that such a probabilistic estimate also holds at all large scales.

4.1. **Fixed disorder.**

Proposition 4.1. *Let $H_{\mathbf{X}}$ be a Poisson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ with density $\varrho > 0$, and fix $p > 0$. Then there exist a constant $C_u > 0$ and a scale $\overline{L}_0 = \overline{L}_0(d, u, \varrho, p) < \infty$, such that for all scales $L \geq \overline{L}_0$ we have (3.26), and, setting*

$$\delta_L = 1 + ((p+d+1)\varrho^{-1} \log L)^{\frac{1}{d}}, \quad E_L = C_u {\delta_L}^{-2(d+1)}, \quad \text{and} \quad m_L = \tfrac{1}{2}\sqrt{E_L}, \tag{4.1}$$

the scale L is (E, m_L)-localizing for all energies $E \in [0, E_L]$.

The proof will be based on the following lemma.

Lemma 4.2. *Let H_X be a Hamiltonian as in (1.1)-(1.3). Given $\delta_0 > 0$ and $L > \delta_0 + \delta_+$, let $\Lambda = \Lambda_L(x)$ and set*

$$J := \{j \in x + \delta_0 \mathbb{Z}^d \cap \Lambda;\ \Lambda_{\delta_L}(j) \subset \widehat{\Lambda})\}, \quad J_e := J \cap (x + 2\delta_0 \mathbb{Z}^d). \tag{4.2}$$

Then there exist constants $C_u > 0$ and $\tilde{\delta}_u \geq \delta_-$, such that if $\delta_0 > \tilde{\delta}_u$, then for all $X, Y \in \mathcal{P}_0(\mathbb{R}^d)$ and $t_Y \in [0,1]^Y$, such that $X \cap Y = \emptyset$ and

$$N_X(\Lambda_{\delta_0}(j)) \geq 1 \quad \text{for all} \quad j \in J_e, \tag{4.3}$$

we have

$$H_{X,(Y,t_Y),\Lambda} \geq 2C_u {\delta_0}^{-2(d+1)} \quad \text{on} \quad \mathrm{L}^2(\Lambda). \tag{4.4}$$

Setting $E_0 = C_u \delta_0^{-2(d+1)}$, it follows that for all $E \in [0, E_0]$ we get

$$\|R_{X,(Y,t_Y),\Lambda}(E)\| \leq E_0^{-1} \tag{4.5}$$

and

$$\|\chi_y R_{X,(Y,t_Y),\Lambda}(E) \chi_{y'}\| \leq 2E_0^{-1} \mathrm{e}^{-\sqrt{E_0}|y-y'|}, \text{ for } y, y' \in \Lambda \text{ with } |y - y'| \geq 4\sqrt{d}. \tag{4.6}$$

Proof. Given configurations X and Y such that $X \cap Y = \emptyset$ and X satisfies (4.3), we pick $\zeta_j \in X_{\Lambda_{\delta_0}(j)}$ for each $j \in J_e$, and set $X_1 := \{\zeta_j,\ j \in J_e\}$, $X_2 = (X \setminus X_1) \sqcup Y$. We claim that for all t_{X_2} we have

$$H_{X_1,(X_2,t_{X_2}),\Lambda} \geq H_{X_1,\Lambda} \geq 2C_u {\delta_0}^{-2(d+1)} \quad \text{on } \mathrm{L}^2(\Lambda), \tag{4.7}$$

where $C_u > 0$. Although the first inequality is obvious, the second is not, since

$$|\{V_{X_1} \neq 0\}| \leq L^d \delta_+^d \delta_0^d < L^d \quad \text{if } \delta_0 > \delta_+. \tag{4.8}$$

To overcome this lack of a strictly positive bound from below for V_{X_1} on Λ, we use the averaging procedure introduced in [BoK]. Requiring $\delta_0 > \delta_-$, we have

$$\overline{V}_{X_1}(y) := \frac{1}{(6\delta_0)^d} \int_{\Lambda_{6\delta_0}(0)} \mathrm{d}a\, V_{X_1}(y-a) \geq c_u\, {\delta_0}^{-d} \chi_\Lambda(y) \quad \text{with } c_u > 0, \tag{4.9}$$

by the definition of X_1 plus the lower bound in (1.3), and hence

$$\overline{H}_{X_1,\Lambda} := -\Delta_\Lambda + \chi_\Lambda \overline{V}_{X_1} \geq c_u {\delta_0}^{-d} \quad \text{on} \quad \mathrm{L}^2(\Lambda). \tag{4.10}$$

Thus, if $\varphi \in C_c^\infty(\Lambda)$ with $\|\varphi\| = 1$, we have

$$\begin{aligned}
\langle \varphi, H_{X_1,\Lambda}\varphi\rangle_\Lambda &= \langle \varphi, \overline{H}_{X_1,\Lambda}\varphi\rangle_\Lambda + \langle \varphi, \left(V_{X_1} - \overline{V}_{X_1}\right)\varphi\rangle_\Lambda \\
&\geq c_u {\delta_0}^{-d} + \langle \varphi, \left(V_{X_1} - \overline{V}_{X_1}\right)\varphi\rangle_{\mathbb{R}^d} \qquad\qquad (4.11)\\
&\geq c_u {\delta_0}^{-d} + \langle \varphi, V_{X_1}\varphi\rangle_{\mathbb{R}^d} - \frac{1}{(6\delta_0)^d}\int_{\Lambda_{6\delta_0}(0)} \mathrm{d}a\, \langle \varphi(\cdot + a), V_{X_1}\varphi(\cdot + a)\rangle \\
&\geq c_u {\delta_0}^{-d} - \frac{1}{(6\delta_0)^d}\int_{\Lambda_{6\delta_0}(0)} \mathrm{d}a\, \left|\langle \varphi, V_{X_1}\varphi\rangle - \langle \varphi(\cdot + a), V_{X_1}\varphi(\cdot + a)\rangle\right| \\
&\geq c_u {\delta_0}^{-d} - c_u' \delta_0 \left\|\nabla_\Lambda \varphi\right\|_\Lambda \geq c_u {\delta_0}^{-d} - c_u'\delta_0 \left(\langle \varphi, H_{X_1,\Lambda}\varphi\rangle\right)_\Lambda^{\frac{1}{2}},
\end{aligned}$$

where we used

$$\|\varphi(\cdot + a) - \varphi\|_{\mathbb{R}^d} = \left\|(\mathrm{e}^{a\cdot\nabla} - 1)\varphi\right\|_{\mathbb{R}^d} \leq |a|\, \|\nabla\varphi\|_{\mathbb{R}^d} = |a|\, \|\nabla_\Lambda\varphi\|_\Lambda\,. \tag{4.12}$$

It follows that there is $\tilde{\delta}_u \geq \delta_-$, such that for $\delta_0 > \tilde{\delta}_u$ we have

$$\langle \varphi, H_{X_1,\Lambda}\varphi\rangle_\Lambda \geq c_u''\, {\delta_0}^{-2(d+1)}, \tag{4.13}$$

and hence we get (4.7), which implies (4.4).

If we now set $E_0 = C_u \delta_0^{-2(d+1)}$, then for all $E \in [0, E_0]$ we get (4.5) immediately from (4.4), and (4.6) follows from (4.4) by the Combes-Thomas estimate (we use the precise estimate in [GK2, Eq. (19)]). ☐

Proof of Proposition 4.1. Given $\varrho > 0$, $p > 0$, let C_u and $\tilde{\delta}_u$ be the constant from Lemma 4.2, and for scales $L > 1$ let δ_L, E_L, and m_L be as in (4.1). Given a box $\Lambda = \Lambda_L(x)$, let J, J_e be as in Lemma 4.2 with $\delta_0 = \delta_L$, and set $\Lambda^{(e)} = \bigcup_{j\in J_e} \Lambda_{\delta_L}(j)$. We require

$$\varrho \leq (p+d+1)\tilde{\delta}_u^{-d} \log L, \quad \text{which implies} \quad \delta_L \geq 1 + \tilde{\delta}_u, \quad \text{and} \quad L > \delta_L + \delta_+. \tag{4.14}$$

We let $\widehat{\mathcal{J}}_\Lambda$ denote the collection of all $(B, B', S) \in \mathcal{J}_\Lambda$ such that

$$B \sqcup B' \sqcup S \in \mathcal{J}_\Lambda, \quad B \sqcup B' \subset \Lambda^{(e)}, \quad S \cap \Lambda^{(e)} = \emptyset; \tag{4.15}$$

$$N_B(\Lambda_{\delta_L}(j)) \geq 1 \quad \text{for all} \quad j \in J_e; \tag{4.16}$$

$$N_S(\Lambda_{\delta_L}(j)) \geq 1 \quad \text{for all} \quad j \in J \setminus J_e. \tag{4.17}$$

If $(B,B',S) \in \widehat{\mathcal{J}}_\Lambda$, it is a consequence of (4.17) that the density condition (3.39) holds for S in Λ if

$$\varrho \geq c_{p,d} L^{-(0+)}, \quad \text{where} \quad c_{p,d} > 0, \tag{4.18}$$

and then it follows from (4.16) and Lemma 4.2 that $\mathcal{C}_{\Lambda,B,B',S}$ is a (Λ, E, m_L)-adapted bevent for all $E \in [0, E_L]$ if we also have

$$\varrho \geq c_{p,d,u} L^{-\frac{d}{d+3}}, \quad \text{where} \quad c_{p,d,u} > 0. \tag{4.19}$$

Moreover, if $(B_i, B_i', S_i) \in \widehat{\mathcal{J}}_\Lambda$, $i = 1, 2$, and $(B_1, B_1', S_1) \neq (B_2, B_2', S_2)$, then $\mathcal{C}_{\Lambda,B_1,B_1',S_1} \cap \mathcal{C}_{\Lambda,B_2,B_2',S_2} = \emptyset$. We conclude that

$$\Omega_\Lambda = \bigsqcup_{(B,B',S)\in\widehat{\mathcal{J}}_\Lambda} \mathcal{C}_{\Lambda,B,B',S} \tag{4.20}$$

is a (Λ, E, m_L)-localizing event $E \in [0, E_L]$ if (4.14), (4.18) and (4.19) are satisfied, which can be assured by requiring that $L > \overline{L}_1(d, u, \varrho, p)$.

To establish (3.57), let $\delta_L' := \delta_L - 1 = ((p + d + 1)\varrho^{-1} \log L)^{\frac{1}{d}}$, and consider the event

$$\Omega_\Lambda^{(\ddagger)} := \{N_{\mathbf{X}}(\Lambda_{\delta_L'}(j)) \geq 1 \quad \text{for all} \quad j \in J\}. \tag{4.21}$$

Clearly

$$\mathbb{P}\{\Omega_\Lambda^{(\ddagger)}\} \geq 1 - \left(\tfrac{L}{\delta_L}\right)^d \mathrm{e}^{-\varrho(\delta_L')^d} \geq 1 - L^{-p-1}. \tag{4.22}$$

Since $\delta_L - \delta_L' = 1 \geq \eta_L$, we must have

$$\Omega_\Lambda^{(\ddagger)} \cap \Omega_\Lambda^{(0)} \subset \Omega_\Lambda, \tag{4.23}$$

and hence (3.57) follows from (4.22) and (3.27) for $L > \overline{L}_0(d, u, \varrho, p)$ satisfying (3.26). □

4.2. **Fixed interval at the bottom of the spectrum and high disorder.** Proposition 4.1 can also be formulated for a fixed interval at the bottom of the spectrum and high disorder.

Proposition 4.3. *Let $H_{\mathbf{X}}$ be a Poisson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ with density $\varrho > 0$, and fix $p > 0$. Given $E_0 > 0$, there exist a constant $C_{d,u,p,E_0} > 0$ and a scale $\overline{L}_0 = \overline{L}_0(d, u, E_0, p) < \infty$, such that if $L \geq \overline{L}_0$ and $\varrho \geq C_{d,u,p,E_0} \log L$ satisfy (3.26), setting $m = \frac{1}{2}\sqrt{E_0}$, the scale L is (E, m)-localizing for all energies $E \in [0, E_0]$.*

Proof. Given $E_0 > 0$ and $p > 0$, let $K_0 = \min\{k \in \mathbb{N};\ k \geq 2u_-^{-1} E_0\}$, $\Lambda = \Lambda_L(x)$, fix $\delta_0 = \frac{1}{6}\delta_-$, and let $J, J_e, \Lambda^{(e)}$ be as in Proposition 4.1 (with δ_0 instead of δ_L). Given $X, Y \in \mathcal{P}_0(\mathbb{R}^d)$ and $t_Y \in [0,1]^Y$, such that $X \cap Y = \emptyset$ and

$$N_X(\Lambda_{\delta_0}(j)) \geq K_0 \quad \text{for all} \quad j \in J_e, \tag{4.24}$$

we have

$$H_{X,(Y,t_Y),\Lambda} \geq 2E_0 \quad \text{on} \quad \mathrm{L}^2(\Lambda), \tag{4.25}$$

and (4.5) and (4.6) follows as in Lemma 4.2.

To prove (4.25), fix $X_1 \subset X$ such that has exactly K_0 points in each box $\Lambda_{\delta_0}(j)$ for all $j \in J_e$ and none outside these boxes, that is,

$$N_{X_1}(\Lambda_{\delta_0}(j)) = K_0 \text{ for all } j \in J_e \quad \text{and} \quad N_{X_1}(\mathbb{R}^d \setminus \Lambda^{(e)}) = 0. \tag{4.26}$$

By our choice of δ_0 and (1.3) we get

$$V_{X_1}(y) \geq K_0 u_- \chi_\Lambda(y) \geq 2E_0 \chi_\Lambda(y), \tag{4.27}$$

and hence, setting $X_2 = X \setminus X_1$, for all $t_{X_2} \in [0,1]^{X_2}$ we have

$$H_{X_1,(X_2,t_{X_2}),\Lambda} \geq H_{X_1,\Lambda} \geq 2E_0, \tag{4.28}$$

and (4.25) follows.

We now modify the argument in the proof of Proposition 4.1. Let $\widehat{\mathcal{J}}_\Lambda$ denote the collection of all $(B,B',S) \in \mathcal{J}_\Lambda$ such that

$$B \sqcup B' \sqcup S \in \mathcal{J}_\Lambda, \quad B \sqcup B' \subset \Lambda^{(e)}, \quad S \cap \Lambda^{(e)} = \emptyset; \tag{4.29}$$

$$N_B(\Lambda_{\delta_0}(j)) \geq K_0 \quad \text{for all} \quad j \in J_e; \tag{4.30}$$

$$N_S(\Lambda_{\delta_0}(j)) \geq 1 \quad \text{for all} \quad j \in J \setminus J_e. \tag{4.31}$$

If $(B,B',S) \in \widehat{\mathcal{J}}_\Lambda$, the density condition (3.39) for S in Λ follows from (4.31), and it follows from (4.30) and (4.25) that $\mathcal{C}_{\Lambda,B,B',S}$ is a (Λ,E,m)-adapted bevent with $m = \frac{1}{2}\sqrt{E_0}$ for all $E \in [0,E_0]$ if $L \geq \overline{L}_1(u,E_0)$. We conclude that

$$\Omega_\Lambda = \bigsqcup_{(B,B',S)\in\widehat{\mathcal{J}}_\Lambda} \mathcal{C}_{\Lambda,B,B',S} \tag{4.32}$$

is a (Λ,E,m)-localizing event for all $E \in [0,E_0]$.

To establish (3.57), let $\delta_1 := \frac{1}{2}\delta_0$ and consider the event

$$\Omega_\Lambda^{(\ddagger)} := \{N_{\mathbf{X}}(\Lambda_{\delta_1}(j)) \geq K_0 \quad \text{for all} \quad j \in J\}. \tag{4.33}$$

We have, using (2.9),

$$\mathbb{P}\{\Omega_\Lambda^{(\ddagger)}\} \geq 1 - \left(\tfrac{L}{\delta_0}\right)^d C_{K_0} \mathrm{e}^{-\frac{1}{2}\varrho\delta_1^d} = 1 - C_{u,E_0,d}L^d \mathrm{e}^{-c_{u,d}\varrho} \geq 1 - L^{-p-1} \tag{4.34}$$

for $\varrho \geq C_{d,u,p,E_0}\log L$ if $L \geq \overline{L}_2(u,E_0,d,p)$

Since $\delta_0 - \delta_1 = \frac{1}{12}\delta_- \geq \eta_L$ for $L \geq \overline{L}_3(u)$, for $L \geq \overline{L}_4(u,E_0,d,p)$ we must have

$$\Omega_\Lambda^{(\ddagger)} \cap \Omega_\Lambda^{(0)} \subset \Omega_\Lambda, \tag{4.35}$$

and hence (3.57) follows from (4.34) and (3.27) for $L > \overline{L}_0(d,u,E_0,p)$ with $\varrho \geq C_{d,u,p,E_0}\log L$. ∎

5. The multiscale analysis with a Wegner estimate

We can now state our version of [BoK, Proposition A′] for Poisson Hamiltonians.

Proposition 5.1. *Let $H_{\mathbf{X}}$ be a Poisson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ with density $\varrho > 0$. Fix an energy $E_0 > 0$. Pick $p = \frac{3}{8}d-$, $\rho_1 = \frac{3}{4}-$ and $\rho_2 = 0+$, more precisely, pick p, ρ_1, ρ_2 such that*

$$\tfrac{8}{11} < \tfrac{d}{d+p} < \rho_1 < \tfrac{3}{4}, \qquad \rho_2 = \rho_1^{n_1} \ \textit{with}\ n_1 \in \mathbb{N} \ \textit{and}\ p < d(\tfrac{\rho_1}{2} - \rho_2). \tag{5.1}$$

Let $E \in [0,E_0]$, and suppose L is (E,m_0)-localizing for all $L \in [L_0^{\rho_1\rho_2}, L_0^{\rho_1}]$, where

$$m_0 \geq L_0^{-\tau_0} \quad \textit{with} \quad \tau_0 = 0+ < \rho_2, \tag{5.2}$$

the condition (3.26) is satisfied at scale $L_0^{\rho_1\rho_2}$, and the scale L_0 is also sufficiently large (depending on $d, E_0, p, \rho_1, \rho_2, \tau_0$) . Then L is $(E, \frac{m_0}{2})$-localizing for all $L \geq L_0$ (actually, for all $L \geq L_0^{\rho_1\rho_2}$).

The proof will require several lemmas and definitions.

Lemma 5.2. *Fix $p' = p-$ and let $\Lambda_\ell \subset \Lambda = \Lambda_L$ with $\ell \ll L$. If the scale ℓ is (E,m)-localizing, then there exists a $(\Lambda, \Lambda_\ell, E, m)$-localized event $\Omega_\Lambda^{\Lambda_\ell}$, i.e.,*

$$\Omega_\Lambda^{\Lambda_\ell} = \bigsqcup_{i=1}^{I_{L,\ell}} \mathcal{C}^{\Lambda_\ell}_{\Lambda, B_i, B_i', S_i} \tag{5.3}$$

for some disjoint $(\Lambda, \Lambda_\ell, E, m)$-adapted bevents $\{\mathcal{C}^{\Lambda_\ell}_{\Lambda, B_i, B_i', S_i}\}_{i=1,2,\ldots,I_{L,\ell}}$, such that

$$\mathbb{P}\{\Omega_\Lambda^{\Lambda_\ell}\} > 1 - \ell^{-p'}. \tag{5.4}$$

Proof. Given disjoint Λ_ℓ-bevents, the corresponding (Λ, Λ_ℓ)-bevents in (3.54) are also disjoint events. Since the scale ℓ is (E,m)-localizing, there is a (Λ_ℓ, E, m)-localized event Ω_{Λ_ℓ} satisfying (3.57). From Lemma 3.13 we get

$$\Omega_{\Lambda_\ell} \cap \Omega_\Lambda^{(0)} \subset \Omega_\Lambda^{\Lambda_\ell}, \tag{5.5}$$

where $\Omega_\Lambda^{\Lambda_\ell}$ is as in (5.3). The estimate (5.4) then follows from (3.57) and (3.27). □

Definition 5.3. *Given scales $\ell \le L$, a standard ℓ-covering of a box $\Lambda_L(x)$ is a collection of boxes Λ_ℓ of the form*

$$\mathcal{G}^{(\ell)}_{\Lambda_L(x)} = \{\Lambda_\ell(r)\}_{r \in \mathbb{G}^{(\ell)}_{\Lambda_L(x)}}, \tag{5.6}$$

where

$$\mathbb{G}^{(\ell)}_{\Lambda_L(x)} := \{x + \alpha\ell\mathbb{Z}^d\} \cap \Lambda_L(x) \quad \textit{with} \quad \alpha \in]\tfrac{3}{5}, \tfrac{4}{5}] \cap \{\tfrac{L-\ell}{2\ell n};\, n \in \mathbb{N}\}. \tag{5.7}$$

Lemma 5.4. *If $\ell \ll L$ there is always a standard ℓ-covering $\mathcal{G}^{(\ell)}_{\Lambda_L(x)}$ of a box $\Lambda_L(x)$, and we have*

$$\Lambda_L(x) = \bigcup_{r \in \mathbb{G}^{(\ell)}_{\Lambda_L(x)}} \Lambda_\ell(r), \tag{5.8}$$

$$\textit{for each } y \in \Lambda_L(x) \textit{ there is } r \in \mathbb{G}^{(\ell)}_{\Lambda_L(x)} \textit{ with } \Lambda_{\frac{2\ell}{5}}(y) \cap \Lambda_L(x) \subset \Lambda_\ell(r), \tag{5.9}$$

$$\Lambda_{\frac{\ell}{5}}(r) \cap \Lambda_\ell(r') = \emptyset \quad \textit{if} \quad r \neq r', \tag{5.10}$$

$$\#\mathbb{G}^{(\ell)}_{\Lambda_L(x)} \le \left(\tfrac{5}{3}\tfrac{L}{\ell}\right)^d \le \left(\tfrac{2L}{\ell}\right)^d. \tag{5.11}$$

Moreover we have the following nesting property: Given $y \in x + \alpha\ell\mathbb{Z}^d$ and $n \in \mathbb{N}$ such that $\Lambda_{(2n\alpha+1)\ell}(y) \subset \Lambda$, it follows that

$$\Lambda_{(2n\alpha+1)\ell}(y) = \bigcup_{r \in \{x+\alpha\ell\mathbb{Z}^d\} \cap \Lambda_{(2n\alpha+1)\ell}(y)} \Lambda_\ell(r), \tag{5.12}$$

and $\{\Lambda_\ell(r)\}_{r \in \{x+\alpha\ell\mathbb{Z}^d\} \cap \Lambda_{(2n\alpha+1)\ell}(y)}$ is a standard ℓ-covering of the box $\Lambda_{(2n\alpha+1)\ell}(y)$.

Proof. The lemma can be easily checked using (5.7). In particular, $\alpha > \frac{3}{5}$ ensures (5.9), $\alpha \le \frac{4}{5}$ ensures (5.10), and the existence of $n \in \mathbb{N}$ such that $2n\alpha\ell = L - \ell$ ensures the nesting property (5.8). □

In the following we fix $E \in [0, E_0]$, assume (5.1), and set $\Lambda = \Lambda_L$, $\ell_1 = L^{\rho_1}$, and $\ell_2 = L^{\rho_2}$. We also assume the induction hypotheses: for each box $\Lambda_\ell \subset \Lambda$ with $\ell \in [\ell_2, \ell_1]$ there is a (Λ_ℓ, E, m_0)-localized event Ω_{Λ_ℓ} with (3.57), and hence it follows from Lemma 5.2 that there is a $(\Lambda, \Lambda_\ell, E, m_0)$-localized event $\Omega_\Lambda^{\Lambda_\ell}$ with (5.4), and we have (5.2) with m_0 and L.

Remark 5.5. *The rate of decay m in* (3.9)*, which by hypothesis is m_0 as in* (5.2) *for all scales $L \in [L_0^{\rho_1\rho_2}, L_0^{\rho_1}]$, will vary along the multiscale analysis, i.e., the construction gives a rate of decay m_L at scale L. The control of this variation can be done as usual, as commented in* [BoK] *(but we need a condition like* (5.2)*), so we always have $m_L \geq \frac{m_0}{2}$, e.g.,* [DrK, FK, GK1, Kl]*). We will ignore this variation as in* [BoK] *and simply write m for m_L. We will omit m from the notation in the rest of this section. The exponent* $1-$ *in* (3.8) *does not vary.*

We now define an event that incorporates [BoK, property $(*)$].

Definition 5.6. *Given a box Λ_{ℓ_1}, for each $n = 0, 1, \ldots, n_1$ let $L_n =: \ell_1^{\rho_1^n}$ (note $L_0 = \ell_1$, $L_{n_1} = \ell_2$), and let $\mathcal{R}_n = \{\Lambda_{L_n}(r)\}_{r\in R_n}$ be a standard L_n-covering of Λ_{ℓ_1} as in* (5.6)*. For a given number K_2, a configuration set X is said to be (Λ_{ℓ_1}, E)-notsobad if there is $\Upsilon_B = \cup_{r\in R'_{n_1}} \Lambda_{3\ell_2}(r)$, where $R'_{n_1} \subset R_{n_1}$ with $\#R'_{n_1} \leq K_2$, such that for all $x \in \Lambda_{\ell_1} \setminus \Upsilon_B$ there is an (X, E)-jgood box $\Lambda_{L_n}(r)$, with $r \in R_n$ for some $n \in \{1, \ldots, n_1\}$ and $\Lambda(x, \frac{2L_n}{5}) \cap \Lambda_{\ell_1} \subset \Lambda_{L_n}(r)$. If $\Lambda_{\ell_1} \subset \Lambda$, a $(\Lambda, \Lambda_{\ell_1})$-bconfset $C_{\Lambda,B}^{\Lambda_{\ell_1}}$ or bevent $C_{\Lambda,B,B'}^{\Lambda_{\ell_1}}$ is $(\Lambda, \Lambda_{\ell_1}, E)$-notsobad if the configuration set B is (Λ_{ℓ_1}, E)-notsobad.*

Lemma 5.7. *For sufficiently large K_2, depending only on d, p, ρ_1, n_1, for all boxes $\Lambda_{\ell_1} \subset \Lambda$, with ℓ_1 large enough, there exist disjoint $(\Lambda, \Lambda_{\ell_1}, E)$-notsobad bevents $\{\mathcal{C}_{\Lambda,B_m,B'_m}^{\Lambda_{\ell_1}}\}_{m=1,2,\ldots,M}$ such that*

$$\mathbb{P}\{\Omega_\Lambda^{\Lambda_{\ell_1},(*)}\} > 1 - \ell_1^{-5d}, \qquad \textit{with} \qquad \Omega_\Lambda^{\Lambda_{\ell_1},(*)} = \bigsqcup_{m=1}^{M} \mathcal{C}_{\Lambda,B_m,B'_m}^{\Lambda_{\ell_1}}, \tag{5.13}$$

and hence

$$\Omega_\Lambda^{\Lambda_{\ell_1},(*\backslash)} := \Omega_\Lambda^{\Lambda_{\ell_1},(*)} \setminus \Omega_\Lambda^{\Lambda_{\ell_1}} = \bigsqcup_{q=1}^{Q} \mathcal{C}_{\Lambda,F_q,F'_q}^{\Lambda_{\ell_1}}, \tag{5.14}$$

where $\{\mathcal{C}_{\Lambda,F_q,F'_q}^{\Lambda_{\ell_1}}\}_{q=1,2,\ldots,Q}$ are disjoint $(\Lambda, \Lambda_{\ell_1}, E)$-notsobad bevents.

Proof. Given $\Lambda_{L_{n-1}}(r) \in \mathcal{R}_{n-1}$, we set

$$\begin{aligned} &\mathcal{R}_n(r) := \{\Lambda_{L_n}(s) \in \mathcal{R}_n;\ \Lambda_{L_n}(s) \cap \Lambda_{L_{n-1}}(r) \neq \emptyset\} \quad \text{and} \\ &R_n(r) := \{s \in \mathcal{R}_n;\ \Lambda_{L_n}(s) \in \mathcal{R}_n(r)\}. \end{aligned} \tag{5.15}$$

We have $\Lambda_{L_{n-1}}(r) \subset \bigcup_{s\in R_n(r)} \Lambda_{L_n}(s)$ and, similarly to (5.11), $\#R_n(r) \leq (\frac{3L_{n-1}}{L_n})^d$. Fix a number K', and define the event $\Omega_\Lambda^{\Lambda_{\ell_1},(*\prime)}$ as consisting of $\omega \in \Omega$ such that, for all $n = 1, \ldots, n_1$ and all $r \in R_{n-1}$, we have $\omega \in \Omega_\Lambda^{\Lambda_{L_n}(s)}$ for all $s \in R_n(r)$, with the possible exception of at most K' disjoint boxes $\Lambda_{L_n}(s)$ with $s \in R_n(r)$. The probability of its complementary event can be estimated from (5.4) as in [BoK, Eq. (6.12)]:

$$\begin{aligned} \mathbb{P}\left\{\Omega \setminus \Omega_{\Lambda_{\ell_1}}^{(*\prime)}\right\} &\leq \sum_{n=1}^{n_1} (\tfrac{2\ell_1}{L_{n-1}})^d (\tfrac{3L_{n-1}}{L_n})^{K'd} L_n^{-K'p'} \\ &\leq 2^d 3^{K'd} n_1 \ell_1^{-\rho_1^{-1}(K'(\rho_1(p'+d)-d)+d(K'-1))} \leq \ell_1^{-6d}, \end{aligned} \tag{5.16}$$

which holds for all large ℓ_1 after choosing K' sufficiently large using (5.1).

Given $\omega \in \Omega_\Lambda^{\Lambda_{\ell_1},(*\prime)}$, then for each $n = 1, \ldots, n_1$ and $r \in R_{n-1}$ we can find $s_1, s_2, \ldots, s_{K''} \in R_n(r)$, with $K'' \leq K' - 1$, such that $\omega \in \Omega_\Lambda^{\Lambda_{L_n}(s)}$ if $s \in R_{n(r)}$ and $s \notin \bigcup_{j=1}^{K''} \Lambda_{3L_n}(s_j)$. (Here we need boxes of side $3L_n$ because we only ruled out the existence of K' *disjoint* boxes of side L_n.) Since each box $\Lambda_{3L_n}(s_j)$ is contained in the union of at most C'' boxes in $\mathcal{R}_n$, we conclude that for each $\omega \in \Omega_\Lambda^{\Lambda_{\ell_1},(*\prime)}$ there are $t_1, t_2, \ldots, t_{K'''} \in R_{n_1}$, with $K''' \leq K_2 = (C''(K'-1))^{n_1}$, such that , setting $\Upsilon = \bigcup_{t_j=1}^{K'''} \Lambda_{3\ell_2}(t_j)$, for all $x \in \Lambda_{\ell_1} \setminus \Upsilon$ we have $\omega \in \Omega_\Lambda^{\Lambda_{L_n}(s)}$ for some $n = 1, 2, \ldots, n_1$ and $s \in R_n$, with and $\Lambda(x, \frac{2L_n}{5}) \cap \Lambda_{\ell_1} \subset \Lambda_{L_n}(s)$.

Recalling (3.46), we have

$$\Omega_\Lambda^{\Lambda_{\ell_1},(*\prime)} \cap \Omega_\Lambda^{(0)} \subset \Omega_\Lambda^{\Lambda_{\ell_1},(*)} := \bigsqcup_{\{(F,F');\, F\sqcup F' \in \mathcal{J}_\Lambda^{\Lambda_{\ell_1}},\, \mathcal{C}_{\Lambda,F,F'}^{\Lambda_{\ell_1}} \cap \Omega_\Lambda^{\Lambda_{\ell_1},(*\prime)} \neq \emptyset\}} \mathcal{C}_{\Lambda,F,F'}^{\Lambda_{\ell_1}}. \tag{5.17}$$

It follows from Lemma 3.6 that each $\mathcal{C}_{\Lambda,F,F'}$ in the disjoint union must be a $(\Lambda, \Lambda_{\ell_1}, E)$-notsobad bevent. Thus (5.13) follows from (5.16) and (3.27). We obtain (5.14) from (5.13) and (3.56). □

Definition 5.8. *Let $\mathcal{R} = \{\Lambda_{\ell_1}(r)\}_{r \in R}$ be a standard ℓ_1-covering of Λ and fix $K_1 \in \mathbb{N}$. A Λ-bevent $\mathcal{C}_{\Lambda,B,B',S}$, is called (Λ, E)-prepared if S satisfies the density condition*

$$\#(S \cap \widehat{\Lambda}_\ell) \geq \ell^{d-}, \quad \textit{for all boxes} \quad \Lambda_\ell \subset \Lambda \quad \textit{with} \quad \ell_1 \ll \ell \leq L, \tag{5.18}$$

and there is $R' \subset R$ with $\#(R \setminus R') \leq K_1$, such that if $r \in R'$ then $\mathcal{C}_{\Lambda,B,B',S}^{\Lambda_{\ell_1}(r)}$ is a $(\Lambda, \Lambda_{\ell_1}(r), E)$-adapted bevent, and if $r \in R \setminus R'$ then $S \cap \Lambda_{\ell_1}(r) = \emptyset$ and $\mathcal{C}_{\Lambda,B,B'}^{\Lambda_{\ell_1}(r)}$ is a $(\Lambda, \Lambda_{\ell_1}(r), E)$-notsobad bevent.

Lemma 5.9. *Let $\mathcal{R} = \{\Lambda_{\ell_1}(r)\}_{r \in R}$ be a standard ℓ_1-covering of Λ. For sufficiently large K_1, depending only on d, p, ρ_1, n_1, if L is taken large enough, there exist disjoint (Λ, E)-prepared bevents $\{\mathcal{C}_{\Lambda,B_m,B'_m,S_m}\}_{m=1,2,\ldots,M_\Lambda}$, such that*

$$\mathbb{P}\{\Omega_\Lambda^{(1)}\} > 1 - 2L^{-2d}, \qquad \textit{with} \qquad \Omega_\Lambda^{(1)} = \bigsqcup_{m=1}^{M_\Lambda} \mathcal{C}_{\Lambda,B_m,B'_m,S_m}. \tag{5.19}$$

Proof. Fix K_1, recall (5.3) and (5.14), and define the event $\Omega_\Lambda^{(1)}$ by the disjoint union

$$\begin{gathered} \Omega_\Lambda^{(1)} := \bigsqcup_{\substack{R' \subset R \\ \#(R \setminus R') \leq K_1}} \Omega_\Lambda^{(1)}(R'), \quad \text{where} \\ \Omega_\Lambda^{(1)}(R') = \left\{ \bigcap_{r \in R'} \Omega_\Lambda^{\Lambda_{\ell_1}(r)} \right\} \bigcap \left\{ \bigcap_{r \in R \setminus R'} \Omega_\Lambda^{\Lambda_{\ell_1}(r),(*\backslash)} \right\}. \end{gathered} \tag{5.20}$$

Using the probability estimates in (5.3) and (5.13), and taking K_1 sufficiently large (independently of the scale), we get

$$\mathbb{P}\{\Omega_\Lambda^{(1)}\} > 1 - 2L^{-2d}, \quad \text{for all} \quad j = 1, 2, \ldots, 2K_1. \tag{5.21}$$

This can be seen as follows. First, from (5.13) and (5.14) we have

$$\mathbb{P}\left\{\Omega_\Lambda^{\Lambda_{\ell_1}(r)} \cup \Omega_\Lambda^{\Lambda_{\ell_1}(r),(*\backslash)}\right\} \geq \mathbb{P}\left\{\Omega_\Lambda^{\Lambda_{\ell_1}(r),(*)}\right\} > 1 - L^{-5\rho_1 d}, \tag{5.22}$$

and hence

$$\begin{aligned}\mathbb{P}\left\{\bigcap_{r\in R}\left\{\Omega_{\Lambda}^{\Lambda_{\ell_1}(r)}\cup\Omega_{\Lambda}^{\Lambda_{\ell_1}(r),(*\backslash)}\right\}\right\} &> 1-\left(\frac{2L}{\ell_1}\right)^d L^{-5\rho_1 d} \\ &\geq 1-2^d L^{-(6\rho_1-1)d} > 1-L^{-2d},\end{aligned} \tag{5.23}$$

for large L, using also (5.1). On the other hand, letting $K_1 = C'(K'-1)$, it follows from (5.3) and (5.1) that

$$\begin{aligned}\mathbb{P}&\left\{\text{there are } K' \text{ disjoint boxes } \Lambda_{\ell_1}(r)\in\mathcal{R} \text{ with } \omega\notin\Omega_{\Lambda}^{\Lambda_{\ell_1}(r)}\right\} \\ &\leq \left(\tfrac{2L}{\ell_1}\right)^{dK'}\ell_1^{-p'K'} \leq 2^{dK'}L^{-K'(\rho_1(p'+d)-d)} \leq L^{-2d}\end{aligned} \tag{5.24}$$

if $K_1 > \frac{2dC'}{\rho_1(p'+d)-d}$ and L is large enough. Here C' is chosen such that the complementary has at most K_1 (not necessarily disjoint) boxes $\Lambda_{\ell_1}(r)\in\mathcal{R}$ with $\omega\notin\Omega_{\Lambda}^{\Lambda_{\ell_1}(r)}$. The estimate (5.21) follows from (5.23) and (5.24).

Moreover, it follows from (5.3) and (5.14) that each $\Omega_{\Lambda}^{(1)}(R')$ is a disjoint union of (non-empty) events of the form

$$\mathcal{D}_{R'} = \left\{\bigcap_{r\in R'}\mathcal{C}_{\Lambda,B_r,B'_r,S_r}^{\Lambda_{\ell_1}(r)}\right\}\bigcap\left\{\bigcap_{r\in R\setminus R'}\mathcal{C}_{\Lambda,F_r,F'_r}^{\Lambda_{\ell_1}(r)}\right\}, \tag{5.25}$$

where $\mathcal{C}_{\Lambda,B_r,B'_r,S_r}^{\Lambda_{\ell_1}(r)}$ is a $(\Lambda,\Lambda_{\ell_1}(r),E)$-adapted bevent for each $r\in R'$, and $\mathcal{C}_{\Lambda,F_r,F'_r}^{\Lambda_{\ell_1}(r)}$ is a $(\Lambda,\Lambda_{\ell_1},E)$-notsobad bevent for each $r\in R\setminus R'$.

It remains to show that $\mathcal{D}_{R'}$ can be written as a disjoint union of (Λ,E)-prepared bevents. To do so let, as in [BoK], let

$$S_{R'} := \{s\in\mathbb{J}_\Lambda;\ s\in\Lambda_{\ell_1}(r)\Rightarrow r\in R' \text{ and } s\in S_r\}. \tag{5.26}$$

Since (5.10) yields

$$\bigcup_{r\in R'} S_r\cap\Lambda_{\frac{\ell_1}{5}}(r)\subset S_{R'}, \tag{5.27}$$

and $\#(R\setminus R')\leq K_1$, it follows as in [BoK, Eq. (6.18)] that $S_{R'}$ satisfies the density condition (5.18) in Λ. It follows from (3.48) and (5.26) that we can rewrite the event $\mathcal{D}_{R'}$ in (5.25) as a disjoint union

$$\mathcal{D}_{R'} = \bigsqcup_{j\in J}\mathcal{C}_{\Lambda,A_j,A'_j,S_{R'}}, \tag{5.28}$$

where $\{\mathcal{C}_{\Lambda,A_j,A'_j,S_{R'}}\}_{j\in J}$ are (Λ,E)-prepared bevents. □

We can now prove a Wegner estimate at scale L using [BoK, Lemma 5.1′].

Lemma 5.10. *Let $\mathcal{C}_{\Lambda,B,B',S}$ be a (Λ,E)-prepared bevent, and consider a box $\Lambda_{L_0}\subset\Lambda$ with $L_0=(2n\alpha+1)\ell_1$ for some $n\in N$, $\ell_1\ll L_0\leq L$, such that Λ_{L_0} is constructed as in (5.12) from a standard ℓ_1-covering of Λ. Then, for sufficiently large L there exist disjoint subsets $\{S_i\}_{i=1,2,\ldots,I}$ of $S_0:=S\cap\Lambda_0$, such that*

$$\left\|R_{B\sqcup S_i,\Lambda_{L_0}}(E)\right\| < \mathrm{e}^{C_1 L^{\frac{4}{3}\rho_1}\log L}, \quad \textit{for all} \quad i=1,2,\ldots,I, \tag{5.29}$$

and we have the conditional probability estimate

$$\begin{aligned} &\mathbb{P}\{\,\Omega^{\Lambda_0}_{\Lambda,B,B',S}\Big|\,\mathcal{C}_{\Lambda,B,B',S}\} > 1 - C_2 L^{-d(\frac{\rho_1}{2}-\rho_2)+}, \qquad \textit{with} \\ &\Omega^{\Lambda_0}_{\Lambda,B,B',S} = \bigsqcup_{i=1}^{I} \mathcal{C}_{\Lambda,B\sqcup S_i,B'\sqcup(S_0\setminus S_i),S\setminus S_0}, \end{aligned} \tag{5.30}$$

where the constants C_1, C_2 do not depend on the scale L. In particular, we get

$$\mathbb{P}\left\{\left\{\|R_{\mathbf{X},\Lambda}(E)\| < \mathrm{e}^{C_1 L^{\frac{4}{3}\rho_1}\log L}\right\} \cap \Omega^{(0)}_{\Lambda}\right\} > 1 - L^{-p}. \tag{5.31}$$

Proof. Let $\mathcal{C}_{\Lambda,B,B',S}$ be a (Λ, E)-prepared cylinder event, consider $\Lambda_{L_0} \subset \Lambda$ as above, and set $B_0 = B \cap \Lambda_{L_0}$, $B'_0 = B' \cap \Lambda_{L_0}$, and $S_0 = S \cap \Lambda_{L_0}$. Let

$$H_{\boldsymbol{\varepsilon}_{S_0}} := H_{B,(S,\boldsymbol{\varepsilon}_S),\Lambda_{L_0}} = H_{B_0,(S_0,\boldsymbol{\varepsilon}_{S_0}),\Lambda_{L_0}} = -\Delta_{\Lambda_{L_0}} + V_{B_0} + \sum_{s\in S_0} \boldsymbol{\varepsilon}_s(\omega) u(x-s), \tag{5.32}$$

where $\boldsymbol{\varepsilon}_{S_0} = \{\boldsymbol{\varepsilon}_s\}_{s\in S_0}$ are independent Bernoulli random variables, with $\mathbb{P}_{\boldsymbol{\varepsilon}_{S_0}}$ denoting the corresponding probability measure. All the hypotheses of [BoK, Lemma 5.1′] are satisfied by the random operator $H(\boldsymbol{\varepsilon}_{S_0})$ in the box Λ_{L_0}. In particular it follows from the density condition (5.18) that S_0 is a collection of “free sites ” satisfying the condition in [BoK, Eq. (5.29)] inside the box Λ_{L_0}. (The fact that we have a configuration $B_0 \cup B'_0 \cup S_0 \subset \mathbb{J}_\Lambda$ instead of a subconfiguration of $\mathbb{Z}^d$ is not important; only the density condition [BoK, Eq. (5.29)] and the fact that $\mathcal{C}_{\Lambda,B_0,B'_0,S_0}$ is (Λ_{L_0}, E)-prepared matter, the specific location of the single-site potentials plays no role in the analysis.)

Thus it follows from [BoK, Lemma 5.1′] that (L large)

$$\mathbb{P}_{\boldsymbol{\varepsilon}_{S_0}}\left\{\left\|R_{\boldsymbol{\varepsilon}_{S_0}}(E)\right\| < \mathrm{e}^{C_1 \ell_1^{\frac{4}{3}}\log \ell_1}\right\} > 1 - C_2 \ell_2^d \ell_1^{-\frac{d}{2}+}, \tag{5.33}$$

where the constants C_1, C_2 do not depend on the scale L. In other words, there is a subset $Q \subset \{0,1\}^{S_0}$ such that

$$\begin{aligned} &\mathbb{P}\{\boldsymbol{\varepsilon}_{S_0} \in Q\} > 1 - C_2 \ell_2^d \ell_1^{-\frac{d}{2}+}, \quad \text{and} \\ &\left\|R_{B\cup\mathcal{X}(S_0,\varepsilon_{S_0}),\Lambda_{L_0}}(E)\right\| < \mathrm{e}^{C_1 \ell_1^{\frac{4}{3}}\log \ell_1} \quad \text{for all} \quad \varepsilon_{S_0} \in Q. \end{aligned} \tag{5.34}$$

We now conclude from (5.34), recalling the definitions of ℓ_1 and ℓ_2, that there exist disjoint Λ-bevents $\{\mathcal{C}_{\Lambda,B\sqcup S_i,B'\sqcup(S_0\setminus S_i),S\setminus S_0}\}_{i=1,2,\ldots,I}$, with each $S_i \subset \mathcal{S}_0$, such that we have (5.29) and (5.30).

Since the event $\Omega^{(1)}_\Lambda$ in (5.19) is a disjoint union of such (Λ, E)-prepared bevents, we have, using also Lemma 3.3 as in the derivation of (3.31) (and changing C_1 slightly), that

$$\mathbb{P}\left\{\left\{\|R_{\mathbf{X},\Lambda}(E)\| < \mathrm{e}^{C_1 L^{\frac{4}{3}\rho_1}\log L}\right\} \cap \Omega^{(0)}_{\Lambda}\,\Big|\,\Omega^{(1)}_{\Lambda}\right\} > 1 - C_2 L^{-d(\frac{\rho_1}{2}-\rho_2)+}, \tag{5.35}$$

and hence, using the probability estimate in (5.19), we have

$$\mathbb{P}\left\{\left\{\|R_{\mathbf{X},\Lambda}(E)\| < \mathrm{e}^{2C_1 L^{\frac{4}{3}\rho_1}\log L}\right\} \cap \Omega^{(1)}_{\Lambda}\right\} > 1 - 2C_2 L^{-d(\frac{\rho_1}{2}-\rho_2)+}. \tag{5.36}$$

The desired (5.31) follows using (5.1). $\square$

We are now ready to finish the proof of Proposition 5.1.

Proof of Proposition 5.1. Fix $E \in [0, E_0]$. It suffices to prove that if L' is E-localizing for all $L' \in [L^{\rho_2}, L^{\rho_1}] = [\ell_2, \ell_1]$, and the scale L is sufficiently large, then L is E-localizing.

Let $\mathcal{C}_{\Lambda,B,B',S}$ be a (Λ, E)-prepared bevent, so there is $R' \subset R_0$ with $\#(R_0 \setminus R') \le K_1$, such that if $r \in R'$ then $\mathcal{C}^{\Lambda_{\ell_1}(r)}_{\Lambda,B,B',S}$ is a $(\Lambda, \Lambda_{\ell_1}(r), E)$-adapted bevent, and if $r \in R_0 \setminus R'$ then $S \cap \Lambda_{\ell_1}(r) = \emptyset$ and $\mathcal{C}^{\Lambda_{\ell_1}(r)}_{\Lambda,B,B'}$ is a $(\Lambda, \Lambda_{\ell_1}(r), E)$-notsobad bevent. Recalling (5.12), we pick $n_0 \in \mathbb{N}$ such that $\ell_0 := (2n_0\alpha + 1)\ell_1 \sim L^{1-}$. By geometrical considerations, we can find boxes $\Lambda^{(j)} = \Lambda_{(2m_j n_0\alpha+1)\ell_1}(s_j) \subset \Lambda$, $j = 1, 2, \ldots, J$, where $J \le K_1$, with $m_j \in \{1, 2, \ldots, 2K_1\}$ and $s_j \in \mathbb{G}^{(\ell_1)}_\Lambda$ for each $j = 1, 2, \ldots, J$, such that $\operatorname{dist}(\Lambda^{(j)}, \Lambda^{(j')}) \ge \ell_0$ if $j \ne j'$, and for each $r \in R_0 \setminus R'$ there is $j_r \in \{1, 2, \ldots, J\}$ such that $\Lambda_{\frac{\ell_0}{5}}(r) \cap \Lambda \subset \Lambda^{(j_r)}$.

Since each $\Lambda^{(j)}$ is of the form given in (5.12), we can apply Lemma 5.10 to each $\Lambda^{(j)}$. Since the $\Lambda^{(j)}$ are disjoint, we can use independence of events based in different $\Lambda^{(j)}$'s, and we may apply Lemma 5.10 (or its proof) to all $\Lambda^{(j)}$. Setting $S_0 = \bigcup_{j=1}^J S \cap \Lambda^{(j)}$ and $\tilde{S} = S \setminus S_0$, we conclude that there exist disjoint subsets $\{S_q\}_{q=1,2,\ldots,Q}$ of S_0, such that for each $q = 1, 2, \ldots, Q$ and all $t_{\tilde{S}} \in [0,1]^{\tilde{S}}$ we have

$$\left\| R_{B \sqcup S_q, (\tilde{S}, t_{\tilde{S}}), \Lambda^{(j)}}(E) \right\| < \mathrm{e}^{C_1 L^{\frac{4}{3}\rho_1} \log L}, \quad \text{for all} \quad j = 1, 2 \ldots, J, \tag{5.37}$$

and we have the conditional probability estimate

$$\begin{aligned} &\mathbb{P}\{\, \Omega'_{\Lambda,B,B',S} \big| \, \mathcal{C}_{\Lambda,B,B',S}\} > 1 - 2K_1 C_2 L^{-d(\frac{\rho_1}{2} - \rho_2)+} \qquad \text{with} \\ &\Omega'_{\Lambda,B,B',S} = \bigsqcup_{i=q}^{Q} \mathcal{C}_{\Lambda, B \sqcup S_q, B' \sqcup (S_0 \setminus S_q), \tilde{S}}. \end{aligned} \tag{5.38}$$

By construction, each configuration in $C_{\Lambda, B \sqcup S_q, \tilde{S}}$ satisfies the hypotheses of [BoK, Lemma 2.14] (see also [BoK, (2.22) and (2.23)]), and hence, recalling also (5.1), we can conclude that $C_{\Lambda, B \sqcup S_q, \tilde{S}}$ is a (Λ, E)-good bconfset. Since it is clear that $\tilde{S}$ satisfies the density condition (3.39) in Λ, each $\mathcal{C}_{\Lambda, B \sqcup S_q, B' \sqcup (S_0 \setminus S_q), \tilde{S}}$ is a (Λ, E)-adapted bevent.

Recalling Lemma 5.9 and the event $\Omega^{(1)}_\Lambda$ in (5.19), we conclude the existence of disjoint (Λ, E)-adapted bevents $\{\mathcal{C}_{\Lambda, B_i, B_i', S_i}\}_{i=1,2,\ldots,I}$, and hence of the (Λ, E)-localized event

$$\Omega_\Lambda = \bigsqcup_{i=1}^{I} \mathcal{C}_{\Lambda, B_i, B_i', S_i}, \tag{5.39}$$

such that

$$\mathbb{P}\{\, \Omega_\Lambda | \, \Omega^{(1)}_\Lambda\} > 1 - 2K_1 C_2 L^{-d(\frac{\rho_1}{2} - \rho_2)+}. \tag{5.40}$$

Using the probability estimate in (5.19) and (5.1), we get that

$$\mathbb{P}\{\Omega_\Lambda\} > 1 - L^{-p}, \tag{5.41}$$

and hence the scale L is E-localizing.

Proposition 5.1 is proven. □

6. The proofs of Theorems 1.1 and 1.2

In view of Propositions 4.1, 4.3, and 5.1, Theorems 1.1 and 1.2 are a consequence of the following proposition, whose hypothesis follows from the conclusion of Proposition 5.1. We recall Definition 3.16.

Proposition 6.1. *Fix $p = \frac{3}{8}d-$ and an energy $E_0 > 0$, and suppose there is a scale L_0 and $m > 0$ such that L is (E, m)-jlocalizing for all $L \geq L_0$ and $E \in [0, E_0]$. Then the following holds $\mathbb{P}$-a.e.: The operator $H_{\mathbf{X}}$ has pure point spectrum in $[0, E_0]$ with exponentially localized eigenfunctions (exponential localization) with rate of decay $\frac{m}{2}$, i.e., if ϕ is an eigenfunction of $H_{\mathbf{X}}$ with eigenvalue $E \in [0, E_0]$ we have*

$$\|\chi_x \phi\| \leq C_{\mathbf{X},\phi}\, e^{-\frac{m}{2}|x|}, \quad \text{for all } x \in \mathbb{R}^d. \tag{6.1}$$

Moreover, there exist $\tau > 1$ and $s \in]0, 1[$ such that for eigenfunctions ψ, ϕ (possibly equal) with the same eigenvalue $E \in [0, E_0]$ we have

$$\|\chi_x \psi\|\, \|\chi_y \phi\| \leq C_{\mathbf{X}} \|T^{-1}\psi\| \|T^{-1}\phi\|\, e^{\langle y\rangle^\tau} e^{-|x-y|^s}, \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{6.2}$$

In particular, the eigenvalues of $H_{\mathbf{X}}$ in $[0, E_0]$ have finite multiplicity, and $H_{\mathbf{X}}$ exhibits dynamical localization in $[0, E_0]$, that is, for any $p > 0$ we have

$$\sup_t \|\langle x\rangle^p e^{-itH_{\mathbf{X}}} \chi_{[0,E_0]}(H_{\mathbf{X}})\chi_0\|_2^2 < \infty. \tag{6.3}$$

Proof. The fact that the hypothesis of Proposition 6.1 imply exponential localization in the interval $[0, E_0]$ is proved in [BoK, Section 7]. Although their proof is written for the Bernoulli-Anderson Hamiltonian, it also applies to the Poisson Hamiltonian by proceeding as in the proof of Proposition 5.1. When [BoK, Section 7] states that a box Λ is good at energy E, we should interpret it as the occurrence of the (Λ, E, m)-jlocalized event Ω_Λ as in (3.58), with probability satisfying the estimate (3.59), whose existence is guaranteed by the hypothesis of Proposition 6.1. We should rewrite such an event as in Lemma 5.2 when necessary, with $p' = \frac{3}{8}d- < p$. With these modifications, plus the use of Lemmas 3.6 and 3.8 when necessary, the analysis of [BoK, Section 7] yields exponential localization for Poisson Hamiltonians.

The decay of eigenfunction correlations given in (6.2) follows for the Bernoulli-Anderson Hamiltonian from a careful analysis of [BoK, Section 7] given in [GK5], and hence it also holds for the Poisson Hamiltonian by the same considerations as above. Finite multiplicity and dynamical localization then follow as in [GK5]. □

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References

[AENSS] Aizenman, M., Elgart, A., Naboko, S., Schenker, J., Stolz, G.: Moment analysis for localization in random Schrödinger operators. Inv. Math. **163**, 343-413 (2006)

[B] Bollobás, B.: *Combinatorics.* Cambridge: Cambridge University Press, 1986

[Bo] Bourgain, J.: On localization for lattice Schrödinger operators involving Bernoulli variables. Geometric aspects of functional analysis. Lecture Notes in Math. **1850**, 77-99. Berlin: Springer, 2004

[BoK] Bourgain, J., Kenig, C.: On localization in the continuous Anderson-Bernoulli model in higher dimension, Invent. Math. **161**, 389-426 (2005)

[CKM] Carmona, R., Klein, A., Martinelli, F.: Anderson localization for Bernoulli and other singular potentials. Commun. Math. Phys. **108**, 41-66 (1987)

[CL] Carmona, R, Lacroix, J.: *Spectral theory of random Schrödinger operators*. Boston: Birkhaüser, 1990

[CoH] Combes, J.M., Hislop, P.D.: Localization for some continuous, random Hamiltonians in d-dimension. J. Funct. Anal. **124**, 149-180 (1994)

[CoHK] Combes, J.M., Hislop, P.D., Klopp, F.: Hölder continuity of the integrated density of states for some random operators at all energies. IMRN **4**, 179-209 (2003)

[CoHKN] Combes, J.M., Hislop, P.D., Klopp, F. Nakamura, S.: The Wegner estimate and the integrated density of states for some random operators. Spectral and inverse spectral theory (Goa, 2000), Proc. Indian Acad. Sci. Math. Sci. **112** , 31-53 (2002)

[CoHM] Combes, J.M., Hislop, P.D., Mourre, E.: Spectral averaging, perturbation of singular spectra, and localization. Trans. Amer. Math. Soc. **348**, 4883-4894 (1996)

[CoHN] Combes, J.M., Hislop, P.D., Nakamura, S.: The L^p-theory of the spectral shift function, the Wegner estimate and the integrated density of states for some random operators. Commun. Math. Phys. **218**, 113-130 (2001)

[DV] Donsker, M., Varadhan, S.R.S.: Asymptotics for the Wiener sausage. Comm. Pure Appl. Math. **28**, 525-565 (1975)

[DrK] von Dreifus, H., Klein, A.: A new proof of localization in the Anderson tight binding model. Commun. Math. Phys. **124**, 285-299 (1989)

[FK] Figotin, A., Klein, A.: Localization of classical waves I: Acoustic waves. Commun. Math. Phys. **180**, 439-482 (1996)

[FiLM] Fischer, W., Leschke, H., Müller, P.: Spectral localization by Gaussian random potentials in multi-dimensional continuous space. J. Stat. Phys. **101**, 935-985 (2000)

[FrMSS] Fröhlich, J.: Martinelli, F., Scoppola, E., Spencer, T.: Constructive proof of localization in the Anderson tight binding model. Commun. Math. Phys. **101**, 21-46 (1985)

[FrS] Fröhlich, J., Spencer, T.: Absence of diffusion with Anderson tight binding model for large disorder or low energy. Commun. Math. Phys. **88**, 151-184 (1983)

[GHK1] Germinet, F., Hislop, P., Klein, A.: Localization for the Schrödinger operator with a Poisson random potential. C.R. Acad. Sci. Paris Ser. I **341**, 525-528 (2005)

[GHK2] Germinet, F., Hislop, P., Klein, A.: Localization at low energies for attractive Poisson random Schrödinger operators. Preprint

[GK1] Germinet, F., Klein, A.: Bootstrap Multiscale Analysis and Localization in Random Media. Commun. Math. Phys. **222**, 415-448 (2001)

[GK2] Germinet, F, Klein, A.: Operator kernel estimates for functions of generalized Schrödinger operators. Proc. Amer. Math. Soc. **131**, 911-920 (2003)

[GK3] Germinet, F., Klein, A.: Explicit finite volume criteria for localization in random media and applications. Geom. Funct. Anal. **13**, 1201-1238 (2003)

[GK4] Germinet, F., Klein, A.: New characterizations of the region of complete localization for random Schrödinger operators. J. Stat. Phys. **122**, 73-94 (2006)

[GK5] Germinet, F., Klein, A.: In preparation

[HM] Holden, H., Martinelli, F.: On absence of diffusion near the bottom of the spectrum for a random Schrödinger operator. Commun. Math. Phys. **93**, 197-217 (1984)

[K] Kingman, J.F.C.: *Poisson processes*. New York: The Clarendon Press Oxford University Press, 1993

[Ki] Kirsch, W.: Wegner estimates and Anderson localization for alloy-type potentials. Math. Z. **221**, 507–512 (1996)

[KiM] Kirsch, W., Martinelli, F.: On the ergodic properties of the spectrum of general random operators. J. Reine Angew. Math. **334**, 141-156 (1982)

[KiSS] Kirsch, W., Stollmann, P., Stolz, G.: Localization for random perturbations of periodic Schrödinger operators. Random Oper. Stochastic Equations **6**, 241-268 (1998)

[Kl] Klein, A.: Multiscale analysis and localization of random operators. In *Random Schrödinger operators: methods, results, and perspectives*. Panorama & Synthèse, Société Mathématique de France. To appear

[KlKS] Klein, A., Koines, A., Seifert, M.: Generalized eigenfunctions for waves in inhomogeneous media. J. Funct. Anal. **190**, 255-291 (2002)

[Klo1] Klopp, F.: Localization for semiclassical continuous random Schrödinger operators. II. The random displacement model. Helv. Phys. Acta **66**, 810-841 (1993)

[Klo2] Klopp, F.: Localization for continuous random Schrödinger operators. Commun. Math. Phys. **167**, 553-569 (1995)

[Klo3] Klopp F.: A low concentration asymptotic expansion for the density of states of a random Schrödinger operator with Poisson disorder. J. Funct. Anal. **145** 267–295 (1997)

[Klo4] Klopp F.: Weak disorder localization and Lifshitz tails: continuous Hamiltonians. Ann. I.H.P. **3**, 711-737 (2002)

[KloP] Klopp, F., Pastur, L.: Lifshitz tails for random Schrödinger operators with negative singular Poisson potential. Comm. Math. Phys. **206**, 57-103 (1999)

[LMW] Leschke, H., Müller, P., Warzel, S.: A survey of rigorous results on random Schrödinger operators for amorphous solids. Markov Process. Related Fields **9**, 729-760 (2003)

[LiGP] Lifshits, I.M., Gredeskul, A.G., Pastur, L.A.: *Introduction to the Theory of Disordered Systems.* New York: Wiley-Interscience, 1988

[MS] Martinelli, F., Scoppola, E.: Introduction to the mathematical theory of Anderson localization. Riv. Nuovo Cimento **10**, 1-90 (1987)

[PF] Pastur, L., Figotin, A.: *Spectra of Random and Almost-Periodic Operators.* Heidelberg: Springer-Verlag, 1992

[R] Reiss, R.-D.: A course on point processes. New York: Springer-Verlag, 1993

[Ro] Robbins, H.: A remark on Stirling's formula. Amer. Math. Monthly **62** , 26–29 (1955)

[S] Simon, B.: Schrödinger semi-groups. Bull. Amer. Math. Soc. **7**, 447-526 (1982)

[SW] Simon, B., Wolff, T.: Singular continuum spectrum under rank one perturbations and localization for random Hamiltonians. Commun. Pure. Appl. Math. **39**, 75-90 (1986)

[Sp] Spencer, T.: Localization for random and quasiperiodic potentials. J. Stat. Phys. **51**, 1009-1019 (1988)

[St1] Stollmann, P.: Lifshitz asymptotics via linear coupling of disorder. Math. Phys. Anal. Geom. **2**, 279-289 (1999)

[St2] Stollmann, P.: Wegner estimates and localization for continuum Anderson models with some singular distributions. Arch. Math. (Basel) **75**, 307-311 (2000)

[Sto] Stolz, G.: Localization for random Schrödinger operators with Poisson potential. Ann. Inst. H. Poincaré Phys. Théor. **63** , 297-314 (1995)

[Sz] Sznitman, A.-S.: *Brownian motion, obstacles and random media.* Berlin: Springer-Verlag, 1998

[U] Ueki, N.: Wegner estimates and localization for Gaussian random potentials. Publ. Res. Inst. Math. Sci. **40**, 29-90 (2004)

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