RECENT ADVANCES ABOUT LOCALIZATION IN CONTINUUM RANDOM SCHRÖDINGER OPERATORS WITH AN EXTENSION TO UNDERLYING DELONE SETS

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ABSTRACT. We review recent results on the universal occurence of Anderson localization in continuum random Schrödinger operators, namely localization for any non trivial underlying probability measure. We extend known results to the case where impurities are located on Delone sets. We also recall the recent localization result for Poisson Hamiltonian. A discussion on the Wegner estimate is provided with a comparison between the "usual" estimate and the one derived through Sperner's type argument and (anti)concentration bounds.

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1. Setup and results

1.1. **Setup and results for the Anderson model.** In this note, we consider random Schrödinger operators on $\mathrm{L}^2(\mathbb{R}^d)$ of the type

$$H_{D,\boldsymbol{\omega}} = H_{\boldsymbol{\omega}} := -\Delta + V_{\boldsymbol{\omega}}, \tag{1.1}$$

where Δ is the d-dimensional Laplacian operator, and $V_{\boldsymbol{\omega}}$ is an Anderson-type random potential,

$$V_{\boldsymbol{\omega}}(x) := \sum_{\zeta\in D} \omega_\zeta\, u(x-\zeta), \tag{1.2}$$

where

The author thanks the hospitality of the Pontificia Universidad Católica de Chile where this note has been written.

(I) the single site potential u is a nonnegative bounded measurable function on $\mathbb{R}^d$ with compact support, uniformly bounded away from zero in a neighborhood of the origin, more precisely,

$$u_-\chi_{\Lambda_{\delta_-}(0)} \le u \le u_+\chi_{\Lambda_{\delta_+}(0)} \quad \text{for some constants } u_\pm, \delta_\pm \in]0,\infty[; \tag{1.3}$$

(II) D is a periodic lattice.

(III) $\boldsymbol{\omega} = \{\omega_\zeta\}_{\zeta\in\mathbb{Z}^d}$ is a family of independent identically distributed random variables, whose common probability distribution μ is non-degenerate with bounded support, and satisfies

$$\{0,1\} \in \operatorname{supp}\mu \subset [0,1]. \tag{1.4}$$

To fix notations, the set of realizations of the random variables $\{\omega_\zeta\}_{\zeta\in D}$ is denoted by $\Omega = \Omega_D = [0,1]^D$; $\mathcal{F}$ denotes the σ-algebra generated by the coordinate functions, and $\mathbb{P} = \mathbb{P}_D = \otimes_{\zeta\in D}\,\mu$ is the product measure of the common probability distribution μ of the random variables $\boldsymbol{\omega} = \boldsymbol{\omega}_D = \{\omega_\zeta\}_{\zeta\in D}$. In other words, we work with the probability space $(\Omega,\mathcal{F},\mathbb{P}) = \bigotimes_{\zeta\in D}\left([0,1],\mathcal{B}_{[0,1]},\mu\right)$, where $\mathcal{B}_{[0,1]}$ is the Borel σ-algebra on $[0,1]$. A set $\mathcal{E}\in\mathcal{F}$ will be called an event.

Under assumption (II), that is if D is a lattice, $H_{\boldsymbol{\omega}}$ is a D-ergodic family of random self-adjoint operators. It follows from standard results (cf. [KiMa, Sto2]) that there exists fixed subsets Σ, Σ_{pp}, Σ_{ac} and Σ_{sc} of $\mathbb{R}$ so that the spectrum $\sigma(H_{\boldsymbol{\omega}})$ of $H_{\boldsymbol{\omega}}$, as well as its pure point, absolutely continuous, and singular continuous components, are equal to these fixed sets with probability one.

We shall take advantage of this review to extend some results to the more general setting

(II') $\exists\, 0 < r \le R < \infty$, s.t. D is a (r,R)-Delone set, that is for any cubes Λ_r, Λ_R of respective sizes r, R, $|D\cap\Lambda_r| \le 1$ and $|D\cap\Lambda_R| \ge 1$.

Recall that a lattice is a particular case of a Delone set.

With condition (1.4), the family of operators $H_{\boldsymbol{\omega}}$ is "normalized", so that, by the Borel-Cantelli lemma, assuming (I), (II'), (III),

$$\text{for } \mathbb{P}\,\text{a.e.}\boldsymbol{\omega},\ \sigma(H_{\boldsymbol{\omega}}) = [0,+\infty[. \tag{1.5}$$

Instead of Condition (III) above we may consider the more general situation:

(III') $\exists\, 0 \le a < b < \infty$ s.t. $\{a,b\} \subset \operatorname{supp}\mu \subset [a,b]$.

Assuming (III'), the operator H_ω may be rewriten as

$$H_{\boldsymbol{\omega}} = -\Delta + V_0 + \sum_{\zeta\in D}\omega'_\zeta u'_\zeta \tag{1.6}$$

with

$$V_0 = a\sum_{\zeta\in D} u_\zeta,\ \omega'_\zeta = \frac{\omega_\zeta - a}{b-a},\ \text{and } u'_\zeta = (b-a)u_\zeta \ge 0. \tag{1.7}$$

The picture (1.5) is lost. The infimum of the spectrum is shifted by a constant $E_0 = \inf\sigma(-\Delta + V_0)$, which becomes the almost sure infimum of the spectrum. If (II) and (III) hold, then by ergodicty there exists a set $\Sigma \subset [E_0,\infty[$ that is the almost sure spectrum of H_ω. If we only assume (II'), then this picture is lost.

It will be convenient to work with the sup norm in $\mathbb{R}^d$:

$$\|x\| := \max\{|x_1|,|x_2|,\dots,|x_d|\} \quad \text{for} \quad x = (x_1,x_2,\dots,x_d)\in\mathbb{R}^d. \tag{1.8}$$

Then

$$\Lambda_L(x) := \left\{ y \in \mathbb{R}^d;\ \|y - x\| < \tfrac{L}{2} \right\} = x + \left] -\frac{L}{2}, \frac{L}{2} \right[^d \tag{1.9}$$

denotes the (open) box of side L centered at $x \in \mathbb{R}^d$. By a box Λ_L we mean a box $\Lambda_L(x)$ for some $x \in \mathbb{R}^d$. Given a set B, we write χ_B for its characteristic function. By χ_x we denote the characteristic function of the unit box centered at $x \in \mathbb{R}^d$, i.e., $\chi_x := \chi_{\Lambda_1(x)}$.

We prove localization at the bottom of the spectrum for the Anderson Hamiltonian without any extra hypotheses. We actually prove stronger versions of Anderson localization (pure point spectrum with exponentially decaying eigenfunctions) and dynamical localization (no spreading of wave packets under the time evolution).

Theorem 1.1. *Let $H_{\boldsymbol{\omega}}$ be an Anderson Hamiltonian on $\mathrm{L}^2(\mathbb{R}^d)$ as above with hypotheses (I), (II), (III'). Then there exists $E_0 = E_0(d, u_\pm, \delta_\pm, \mu) > 0$ such that $H_{\boldsymbol{\omega}}$ exhibits Anderson localization as well as dynamical localization in the energy interval $[0, E_0]$. More precisely:*

- *(Anderson localization) There exists $m = m(d, V_{\mathrm{per}}, u_\pm, \delta_\pm) > 0$ such that the following holds with probability one:*
  - *$H_{\boldsymbol{\omega}}$ has pure point spectrum in $[0, E_0]$.*
  - *If ϕ is an eigenfunction of $H_{\boldsymbol{\omega}}$ with eigenvalue $E \in [0, E_0]$, then ϕ is exponentially localized with rate of decay m, more precisely,*

$$\|\chi_x \phi\| \le C_{\boldsymbol{\omega},\phi}\, e^{-m|x|} \qquad \textit{for all} \quad x \in \mathbb{R}^d. \tag{1.10}$$

  - *The eigenvalues of $H_{\boldsymbol{\omega}}$ in $[0, E_0]$ have finite multiplicity.*
- *(Dynamical localization) For all $s < \frac{3}{8}d$ we have*

$$\mathbb{E}\left\{ \sup_{t\in\mathbb{R}} \left\| \langle x \rangle^{\frac{m}{2}} e^{-itH_\omega} \chi_{[0,E_0]}(H_\omega) \chi_0 \right\|_2^{\frac{2s}{m}} \right\} < \infty \quad \textit{for all } m \ge 1. \tag{1.11}$$

The full proof of Theorem 1.1 is presented in [GK4]. In particular it combines the multiscale analysis of Bourgain and Kenig [BK] together with the concentration bound of [AGKW]. This yields Anderson localization (using [GK2] for finite multiplicity). To get dynamical localization, one builds on ideas that are by now standard and that come from [A, DeRJLSi, GDB, G, DSto, GK1, GK2].

1.2. **Extension to underlying Delone sets.** The following theorem extends known results, in particular [BoNStoSt, Theorem 11], where the regularity of the random variable is assumed.

Theorem 1.2. *Conclusions of Theorem 1.1 hold under condtions (I), (II'), (III). Constants then also depend on r, R.*

Remark 1.3. *So far, the general case that would consist in assuming (II') and (III') is out of reach for the multiscale analysis cannot be started with current methods. Indeed if D is not a lattice, both the argument we provide in Section 3 and the Lifshtiz tails approach (e.g.* [Kl3, Kl4, Sto2]*) fail when the bottom of the spectrum is not zero.*

Remark 1.4 (The Bernoulli case). *If the random variables ω_ζ are Bernoulli, taking values $0, 1$ (so that hypothesis (III) holds), then for a given configuration $\boldsymbol{\omega}$, the*

Hamiltonian reads, with $D(\boldsymbol{\omega}) := \{\zeta \in D, \omega_\zeta = 1\}$*,*

$$H_{\boldsymbol{\omega}} = -\Delta + \sum_{\zeta \in D(\boldsymbol{\omega})} u_\zeta. \tag{1.12}$$

One may wonder what can be said about sets $\mathcal{D}(\boldsymbol{\omega})$*'s for which localization is proved. It is clear that they are not* (r, R)*-Delone sets anymore (otherwise the spectrum would not start at zero). However it is interesting to note that as a by product of the proof,* $\mathcal{D}(\boldsymbol{\omega})$ *is relatively dense in the following weak sense: for any* $\varepsilon > 0$*, for any* $x \in \mathbb{R}^d$*,*

$$\lim_{L \to \infty} L^{-(d-\varepsilon)} |\Lambda_L(x) \cap D(\boldsymbol{\omega})| = +\infty. \tag{1.13}$$

This observation follows from the existence of free sites, at any scale large enough, which associated value can be turned to 1 *at the end of the multiscale analysis, ensuring the presence of the point.*

Instead of considering just one underlying Delone set, one may want to look at a family of such sets. A common way of "randomizing" D is to consider the complete metric space given by the closure, with respect to the Delone topology e.g. [LeMoSo], of all its translates:

$$\mathcal{D} = \overline{\{x + D, x \in \mathbb{R}^d\}}. \tag{1.14}$$

Provided D has finite complexity, e.g. [LeMoSo], such a set possesses a Haar measure that we shall denote by ν. It is then possible to consider "thinned" or "coloured" Delone sets $D^{\boldsymbol{\omega}} := (D, \boldsymbol{\omega}_D)$ on $\mathcal{D}$, and to construct the associated Schrödinger operator, which amounts, for any $D \in \mathcal{D}$, to consider the model $H_{D^{\boldsymbol{\omega}}} = H_{D,\boldsymbol{\omega}_D}$ described in (1.1). The probabilistic structure of such a colouring of $\mathcal{D}$ is well described in [MuR], elaborating on [Ho] who considered Bernoulli colourings on Penrose tilings. In particular the overall probability measure $d\hat{\mathbb{P}}(D^{\boldsymbol{\omega}})$ can be decomposed as $d\nu(D) \times d\mathbb{P}_D(\boldsymbol{\omega})$ [MuR, Theorem 3.5]. In particular, this enables one to first perform a conditionning with respect to the Delone variable and conduct the analysis with the random variables.

Since the event of $\{\sigma_c(H_{D,\boldsymbol{\omega}}) = \emptyset\}$ is $\hat{\mathbb{P}}$-measurable, the following statement follows from Theorem 1.2 along the same lines as [AGKW].

Corollary 1.5. *Assume* D *has finite local complexity. There exists* $E(\mu) > 0$*, such that, for* $\hat{\mathbb{P}}$ *a.e.* $\boldsymbol{\omega}$*,* $H_{D^{\boldsymbol{\omega}}}$ *exhibits spectral localization in* $[0, E(\mu)]$*, that is pure point spectrum.*

Remark 1.6. *Extension of Corollary 1.5's result to the localization picture described in Theorem 1.1 and Theorem 1.2 requires a carefull treatment of measurability, since, one has to make sure that events considered throughout the multiscale analysis are jointly measurable in* ν *and* $\mathbb{P}$*, perform the conditionning and do the multiscale analysis. It is very likely that this can be done along the lines of* [GK4].

For pure Delone sets, that is with no random colouring, the situation is much more delicate. One can nevertheless show for large dense sets of Delone sets that localization holds [GMu].

1.3. **The Poisson model.** Writing the (Bernoulli-)Hamiltonian in the form (1.12) is reminiscent to cases where the randomness is introduced by the location of the impurities, rather than by their amplitudes as in the Anderson model. A popular model of such a Schrödinger operator with impurities located at random is given by the Poisson Schrödinger operator, where single site potentials are centered at points obtained through a Poisson point process of a given intensity. While localization in any dimension is expected for many years for this model, at least since the proof of Lifshitz tails provided by Donskher and Varadhan in 1975 [DoV], a rigorous proof of this phenomena has recently been obtained in [GHK1, GHK2] for repulsive potentials and [GHK3] for attractive potential. We review this result in the sequel. Note however that localization in dimension one was known to hold by the work of Stolz [St].

Let us note that another model obtained by randomizing the location of impurities is also of interest: the random displacement model. Only partial results are known for this model: localization in dimension 1 [BuSt, DSiSt], and an asymptotics result (of semi-classical type) in higher dimensions [Kl1]. Other models of insterest have been studied, such as potentials given by Gaussian random variables, see [LeMuW, U].

The Poisson Hamiltonian is the random Schrödinger operator on $\mathrm{L}^2(\mathbb{R}^d)$ given by

$$H_X = -\Delta + V_X, \quad \text{with} \quad V_X(x) = \sum_{\zeta \in X} u(x-\zeta), \tag{1.15}$$

where the single-site potential u is a nonnegative C^1 function on $\mathbb{R}^d$ with compact support satisfying (1.3), and V_X is a Poisson random potential, that is, X is a Poisson process on $\mathbb{R}^d$ with density $\varrho > 0$. Thus the configuration X is a random countable subset of $\mathbb{R}^d$, and, letting $N_X(A)$ denote the number of points of X in the Borel set $A \subset \mathbb{R}^d$, each $N_X(A)$ is a Poisson random variable with mean $\varrho|A|$ (i.e., $\mathbb{P}_\varrho\{N_X(A) = k\} = (\varrho|A|)^k(k!)^{-1}\mathrm{e}^{-\varrho|A|}$ for $k = 0, 1, 2, \ldots$), and the random variables $\{N_X(A_j)\}_{j=1}^n$ are independent for disjoint Borel sets $\{A_j\}_{j=1}^n$. We will denote by $(\mathcal{X}, \mathbb{P}_\varrho)$ the underlying probability space for the Poisson process with density ϱ.

Note that H_X is an ergodic (with respect to translations in $\mathbb{R}^d$) random self-adjoint operator. It follows that the spectrum of H_X is the same for $\mathbb{P}_\varrho$-a.e. X, as well as the decomposition of the spectrum into pure point, absolutely continuous, and singular continuous spectra. For u as above we actually get $\sigma(H_X) = [0, +\infty[$ for $\mathbb{P}_\varrho$-a.e. X [KiMa].

Theorem 1.7. [GHK2] *Given $\varrho > 0$, there exists $E_0 = E_0(\varrho) > 0$ and $m = m(\rho) > 0$, such that conclusions of Theorem 1.1 hold on $[0, E_0]$.*

2. A bit of history and the Wegner estimate

2.1. **Some history.** In the one-dimensional case the continuous Anderson Hamiltonian has been long known to exhibit spectral localization in the whole real line for any non-degenerate μ, i.e. when the random potential is not constant [GoMP, KotSi, DSiSt].

In the multidimensional case, localization at the bottom of the spectrum is already known at great, but nevertheless not all-inclusive, generality; cf. [Sto2, K, BK] and references therein. First proofs of this result are due to Combes Hislop [CH1] and Klopp [Kl2], assuming that the single site probability distribution μ is absolutely continuous with bounded density. The result relies on a multiscale analysis argument "à la" Fröhlich Spencer [FrSp] and adapted from [DrK]'s discrete version; it took more time and a lot of efforts to carry the Aizenman Molchanov approach of fractional moments [AM] over the continuum [AENSSt], still under the regularity assumption on μ.

The absolute continuity condition of μ can be relaxed to Hölder continuity of μ, both in the approach based on the multiscale analysis, and in the one based on the fractional moment method. The basis in the former case is an improved analysis of the Wegner estimate, which was first noticed by Stollmann in [Sto1]. Important improvements in Wegner estimates with (not too) singular continous measures μ have then been successively obtained in [CHNa, CHKl1, CHKlR, GKS, HuKiNaStoV] until the recent optimal form due to Combes Hislop and Klopp [CHKl2]; all theses improved forms provide in particular some continuity property of the integrated density of states.

However, techniques relying on the regularity of μ seem to reach their limit with log-Hölder continuity. In particular, until recently the Bernoulli random potential had been beyond the reach of analysis in more than one dimension. For that extreme case, i.e., of $H_{\boldsymbol{\omega}}$ with $\mu\{1\} = \mu\{0\} = \frac{1}{2}$, localization at the bottom of the spectrum was recently proven by Bourgain and Kenig [BK].

In [BK], the Wegner estimate is obtained along the lines of (an elaborated version of) the multiscale analysis, scale by scale, through a combination of a quantitative unique continuation principle together with a lemma due to Sperner [Sp]. Although it defenitely requires some technical care, it is quite clear from the analysis of [BK] that the result extends to any measure for which a Sperner's type argument is valid. See for an illustration of this point the note [GK3] where μ is a uniform measure on some Cantor set (μ turns to be $\log\log$-Hölder continuous in this example).

Localization was thus proved for the two extreme cases: μ regular enough and μ Bernoulli, and with two different proofs, none of which applying directly to the other case. Our motivation was then to find a single proof for any non degenerated measure, and thus unifying these two extreme results. A key step, the concentration bound extending the Sperner's Lemma estimate used by Bourgain and Kenig, was obtained in [AGKW]. The full technical details of the extension of the multiscale analysis of [BK] are provided in [GK4].

2.2. **The Wegner estimate.** It is easy to understand (or at least to get a hint of) why regularity of the distribution might help for a proof of a Wegner type estimate. But let us first describe what a Wegner type estimate is and what it is good for. The multiscale analysis deals with resolvents of the random Hamiltonians, restricted to finite volume cubes. The aim of the game is to show that such kernels of finite volume resolvents decay exponentially with a good probability. Before showing that resolvents decay exponentially, it sounds reasonable to make sure that their norm is not too big, namely at most sub-exponentially big (so that it does not destroy the exponential decay that has already been obtained from previous scales).

To fix notations, consider a scale L, $H_{L,\boldsymbol{\omega}}$ a suitable restriction of $H_{\boldsymbol{\omega}}$ to a cube Λ_L of side L with Dirichlet boundary condition, and $R_{L,\boldsymbol{\omega}}(z)$ its resolvent (that is now a compact operator). The spectrum of $H_{L,\boldsymbol{\omega}}$ is thus discrete and given $E \in \sigma(H_{\boldsymbol{\omega}}) = [0, +\infty[$ we want to investigate the size of $\|R_{L,\boldsymbol{\omega}}(E)\|$ and show it is $\leq e^{L^{1-\delta}}$, $\delta > 0$, with probability at least $1 - L^{-p}$, for some $p > 0$ (note that $\|R_{L,\boldsymbol{\omega}}(E)\|$ may be infinite, namely when $E \in \sigma(H_{L,\boldsymbol{\omega}})$, but typically, this should happen for a set of $\boldsymbol{\omega}$'s of small measure. This amounts to analyse the probability that $\operatorname{dist}(E, \sigma(H_{L,\boldsymbol{\omega}})) \geq e^{-L^{1-\delta}}$.

The strong form of the Wegner estimate reads as follows [CHKl2]: there exists $C_W < \infty$, such that for η small enough and L large enough,

$$\mathbb{P}(\mathrm{dist}(E, \sigma(H_{L,\boldsymbol{\omega}})) < \eta) \leq C_W Q_{\omega_0}(2\eta) L^d, \tag{2.1}$$

where $Q_{\omega_0}(\eta)$ is the (Levy) concentration function of the random variable ω_0 (or equivalently the modulus of continuity of its measure μ), that is,

$$Q_{\omega_0}(\eta) = \sup_{x\in\mathbb{R}} \mathbb{P}(\omega_0 \in [x, x+\eta[) = \sup_{x\in\mathbb{R}} \mu([x, x+\eta[). \tag{2.2}$$

It is worth pointing out that (2.1) is an a priori estimate that is independent of the existence of localized states. Applying (2.1) with $\eta = e^{-L^{1-\delta}}$ obviously leads to the needed estimate. A weaker version, corresponding to the approach of Bourgain Kenig, reads as follows. Let S be a subset of $D \cap \Lambda_L$, and $\omega_S = (\omega_\zeta)_{\zeta\in S}$. There exists $C_W < \infty$ and $\delta_0 > 0$ s.t., for suitable events $F_{L,\boldsymbol{\omega},S} \subset \mathcal{F}$ coming from the multiscale analysis, for L large enough, $\delta, \varepsilon > 0$ small enough,

$$\mathbb{P}_S(\mathrm{dist}(E, \sigma(H_{L,\boldsymbol{\omega}})) < e^{-L^{1-\delta}}; F_{L,\boldsymbol{\omega},S}) \leq L^\varepsilon Q_Z(2e^{-L^{1-\delta}}), \tag{2.3}$$

where $\mathbb{P}_S = \bigotimes_{\zeta\in S} \mu$ is the restriction of $\mathbb{P}$ to S, $Z = \Phi(\omega_S)$ is a random variable such that for any ω_S, for any $v_\zeta \geq \delta_0$,

$$\Phi(\omega_S + v_\zeta) - \Phi(\omega_S) > 2e^{-L^{1-\delta}}. \tag{2.4}$$

In practice, Φ is an eigenvalue of the finite volume operator, and property (2.4) follows from a quantitative unique continuation principle. Note that unlike what happens in the strong form, it is a only collective effect of the random variables ω_ζ, $\zeta \in S$, that provides some decay. The best universal bound is the following concentration bound (as proven in [AGKW], see Theorem 2.1 below)

$$Q_Z(2e^{-L^{1-\delta}}) \leq C|S|^{-\frac{1}{2}}. \tag{2.5}$$

In practice, $|S| = L^{\frac{3}{4}d-}$, so that the probability in (2.3) amounts to $L^{-\frac{3}{8}d+}$. We shall discuss this point in the next subsection.

One way to understand this difference between regular and singular measures is to consider a purely discrete diagonal model, i.e. where $H_{L,\boldsymbol{\omega}} = V_{L,\boldsymbol{\omega}}$ is a diagonal matrix, with entries labelled by $n = 0, 1, \cdots, N = |\Lambda \cap D|$. Since the eigenvalues are exactly the ω_n's, the distance between E and the spectrum of this diagonal matrix is exactly $\inf_n |E - \omega_n|$. As a consequence

$$\mathbb{P}((\mathrm{dist}(E, \sigma(H_{L,\boldsymbol{\omega}})) < \eta) \leq N\mu(]E-\eta, E+\eta[) \leq Q_{\omega_0}(2\eta)N. \tag{2.6}$$

Note that it is the concentration of a single random variable that enters (2.6). Assume now the measure μ is singular, say Bernoulli with even probability $\frac{1}{2}$, then as soon as $]E-\eta, E+\eta[$ contains an atom of μ, a single ω_n is enough to spoil the picture: we get $\mathbb{P}((\mathrm{dist}(E, \sigma(H_{L,\boldsymbol{\omega}})) < \eta) \geq \frac{1}{2}$ and the situation is desperate! This simple example tell us that 1) some correlation between the eigenvalues is needed (in particular note that if $\Phi(\omega_S) = \omega_1$, then (2.4) fails) and it is the Laplacian and the unique quantitative principle that shall provide this; 2) it is by a collective effect that $\mathbb{P}((\mathrm{dist}(E, \sigma(H_{L,\boldsymbol{\omega}})) < \eta)$ has a chance to be small, and this is typically what Sperner's theorem provides.

2.3. **Antichains, Sperner's lemma and** [AGKW]**'s concentration bound.** The configuration space $\{0,1\}^N$ for a collection of Bernoulli random variables $\boldsymbol{\eta} = \{\eta_1, ..., \eta_N\}$ is partially ordered by the relation defined by:

$$\boldsymbol{\eta} \prec \boldsymbol{\eta}' \quad \Longleftrightarrow \quad \text{for all } \; i \in \{1, ..., N\} : \quad \eta_i \leq \eta_i' \,. \tag{2.7}$$

A set $\mathcal{A} \subset \{0,1\}^N$ is said to be an *antichain* if it does not contain any pair of configurations which are comparable in the sense of "$\prec$". The original Sperner's Lemma [Sp] states that for any such set: $|\mathcal{A}| \leq \binom{N}{[\frac{N}{2}]}$. An immediate computation using Stirling formula shows that the latter is bounded by $C2^N/\sqrt{N}$. A more general result is the LYM inequality for antichains (e.g. [An]):

$$\sum_{\boldsymbol{\eta} \in \mathcal{A}} \frac{1}{\binom{N}{|\boldsymbol{\eta}|}} \; \leq \; 1 \,, \tag{2.8}$$

where $|\boldsymbol{\eta}| = \sum \eta_j$. The LYM inequality has the following probabilistic implication. If $\{\eta_j\}$ are independent copies of a Bernoulli random variable η with probabilities $(1-p, p)$, then for any antichain $\mathcal{A} \subset \{0,1\}^N$:

$$\mathbb{P}\left(\{\boldsymbol{\eta} \in \mathcal{A}\}\right) \; \leq \; \frac{2\sqrt{2}}{\sigma_\eta \sqrt{N}}, \tag{2.9}$$

where $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_N)$, $\sigma_\eta = \sqrt{pq}$ is the standard deviation of η. The same bound extends to antichains on larger alphabet: $\{0, 1, \cdots, k\}^N$ with $k \geq 1$ for equidistributed weights [An] as well as for general weights [En1, En2] (more than an upper bound, an asymptotics as N goes to ∞ is proven in those cases). An extension of (2.9) to independent Bernoulli variables, but no necessarily identically distributed is proven in [AGKW].

Such bounds on probability of antichains find their natural generalization in the following theorem, that deals with arbitrary non degenerate random variables and that is proved in [AGKW].

Theorem 2.1. *Let $\boldsymbol{X} = (X_1, \ldots, X_N)$ be a collection of independent random variables whose distributions satisfy, for all $j \in \{1, ..., N\}$:*

$$\mathbb{P}\left(\{X_j \leq x_-\}\right) \geq p_- \quad \textit{and} \quad \mathbb{P}\left(\{X_j > x_+\}\right) > p_+ \tag{2.10}$$

at some $p_\pm > 0$ and $x_- < x_+$, and $\Phi : \mathbb{R}^N \mapsto \mathbb{R}$ a function such that for some $\varepsilon > 0$

$$\Phi(\boldsymbol{u} + v\boldsymbol{e}_j) - \Phi(\boldsymbol{u}) \; > \; \varepsilon \tag{2.11}$$

for all $v \geq x_+ - x_-$, all $\boldsymbol{u} \in \mathbb{R}^N$, and $j = 1, \ldots, N$, with $\boldsymbol{e}_j$ the unit vector in the j-direction. Then, the random variable Z, defined by $Z = \Phi(\boldsymbol{X})$, obeys the concentration bound

$$Q_Z(\varepsilon) \; \leq \; \frac{4}{\sqrt{N}} \sqrt{\frac{1}{p_+} + \frac{1}{p_-}} \,. \tag{2.12}$$

If the random variables are Bernoulli then the link between Theorem 2.1 and Sperner's theory of antichains is quite obvious. Indeed, let $\varepsilon, \varepsilon'$ be two comparable realizations of $(X_1, \cdots, X_n)$, say $\varepsilon_j \leq \varepsilon_j'$ for all $j = 1, \cdots, N$. Then by (2.11), $\Phi(\varepsilon)$ and $\Phi(\varepsilon')$ cannot both belong to a given interval of length ε. In other words, for any given $x \in \mathbb{R}$, realizations of $Z = \Phi(X_1, \cdots, X_n)$ that fall into $[x, x+\varepsilon[$ belong to an antichain; (2.9) above then yields (2.12).

It remains to extend such a reasonning to arbitrary non degenerate random variables, and not just Bernoulli. This is achieved by taking advantage of a Bernoulli decomposition of random variables described in [AGKW]. This decomposition enables us to rewrite each variable as (in law) $X_i = F_i(t_i) + \delta_i(t_i)\varepsilon_i$, where F_i, δ_i are measurable functions, t_i is a random variable on $]0,1[$ with uniform distribution, ε_i is a Bernoulli random variable independent of t_i. Moreover it is shown in [AGKW] that under condition (2.10), $\mathbb{P}_{t_i}(\delta_i(t_i) \geq x_+ - x_-) \geq p_- + p_+$. A large deviation argument enables us to restrict ourselves to the latter case, that is where $\delta_i(t_i) \geq x_+ - x_-$ for all $i = 1, \cdots, N$. We are thus left with Bernoulli variables for which (2.11) applies (since $\delta_i(t_i) \geq x_+ - x_-$); as before (2.9) finishes the proof.

The Bernoulli decomposition we used here found also an application to random matrices theory [BrG].

3. Proof of Theorem 1.2

With Theorem 2.1 in hands, the Bourgain-Kenig multiscale analysis can be conducted in the same way as in [GK4], provided we can start the algorithm and make sure the density condition on the so called "free sites" is satisfied at all scales. Both points will be clear from the construction we give in Section 3.2 below. It is indeed enough to show that with a sufficiently good probability, the bottom of the spectrum of finite volume operators is lifted up, uniformly with respect to the random variables attached to a set $S \subset D$ s.t. $|S \cap \Lambda| = C_R|\Lambda|$ for some $C_R < \infty$ (actually $(C_R = (2R)^{-d})$.

3.1. **Finite volume operators.**

Given a box $\Lambda = \Lambda_L(x)$ in $\mathbb{R}^d$, we denote by $\hat{\Lambda}$ the subcube $\Lambda_{L-\delta_+}(x)$ (recall (1.3)). We then define finite volume operators as follows:

$$H_{\boldsymbol{\omega},\Lambda} := -\Delta_\Lambda + V_{\boldsymbol{\omega},\Lambda} \quad \text{on} \quad \mathrm{L}^2(\Lambda), \tag{3.1}$$

where Δ_Λ is the Laplacian on Λ with Dirichlet boundary condition, and

$$V_{\boldsymbol{\omega},\Lambda} = \sum_{\zeta \in D \cap \hat{\Lambda}} \omega_\zeta u_\zeta. \tag{3.2}$$

Since we are using Dirichlet boundary condition, we always have $\inf \sigma(H_{\boldsymbol{\omega},\Lambda}) > 0$.

The multiscale analysis estimates probabilities of desired properties of finite volume resolvents at energies $E \in \mathbb{R}$. As in [B, BK, GHK3], these properties include 'free sites'. Given a box Λ, a subset $S \subset \hat{\Lambda}$, and $\mathbf{t}_S = \{t_\zeta\}_{\zeta \in S} \in [0,1]^S$, we set

$$H_{\boldsymbol{\omega},\mathbf{t}_S,\Lambda} := H_{0,\Lambda} + V_{\boldsymbol{\omega},\mathbf{t}_S,\Lambda} \quad \text{on} \quad \mathrm{L}^2(\Lambda), \tag{3.3}$$

where $V_{\boldsymbol{\omega},\mathbf{t}_S,\Lambda} = \chi_\Lambda V_{\boldsymbol{\omega}_\Lambda,\mathbf{t}_S}$ with

$$V_{\boldsymbol{\omega}_\Lambda,\mathbf{t}_S}(x) := V_{\boldsymbol{\omega}_{\Lambda\setminus S}}(x) + V_{\mathbf{t}_S}(x) = \sum_{\zeta \in \hat{\Lambda}\setminus S} \omega_\zeta\, u_\zeta(x-\zeta) + \sum_{\zeta \in S} t_\zeta\, u_\zeta(x-\zeta). \tag{3.4}$$

$R_{\boldsymbol{\omega},\mathbf{t}_S,\Lambda}(z)$ will denote the corresponding finite volume resolvent.

3.2. **Proof of Theorem 1.2.**

Given an energy E, to start the multiscale analysis we will need, as in [B, BK, GHK3], an *a priori* estimate on the probability that a box Λ_L is 'good' with an adequate supply of free sites, for some sufficiently large scale L. The multiscale analysis will then show that such a probabilistic estimate also holds at all large scales.

To prove the needed initial estimate, it is enough to prove that a spectral gap occurs above 0 for finite volume operators with a good probability. This is the purpose of the next proposition.

Proposition 3.1. *Fix $p > 0$ and $0 < \varepsilon \leq 1$. There exists a scale $\tilde{L} = \tilde{L}(d, q, u_-, \delta_-, \mu, p, \varepsilon)$, such that for all scales $L \geq \tilde{L}$ and all $x \in \mathbb{R}^d$ we have*

$$\mathbb{P}\left\{H_{\boldsymbol{\omega},\mathbf{t}_S,\Lambda_L(x)} \geq CR^{-2d-2}(\log L)^{-\frac{2}{d}} \text{ for all } \mathbf{t}_S \in [0,1]^S\right\} \geq 1 - L^{-pd}, \tag{3.5}$$

where $S \subset \Lambda$, $|S| = (2R)^{-d}|\Lambda|$.

Proof. By definition of D being a (r, R)-Delone set, for any $j \in \mathbb{Z}^d$, there exists a point such that $\zeta_j \in D \cap \Lambda_R(j)$ (if ζ_j is not unique, we select one). We define the set $\Upsilon_R \subset D$ to be the collection of these ζ_j's, and we further define Υ_R^0 as the subcollection corresponding respectively to points ζ_j with $j \in (2\mathbb{Z})^d$. Note that $|\Upsilon_R \cap \Lambda| = R^{-d}|\Lambda|$, and $|\Upsilon_R^0 \cap \Lambda| = (2R)^{-d}|\Lambda|$. We set $S = \Upsilon_R^0 \cap \Lambda$.

We further set

$$V_{\Upsilon_R^0,\boldsymbol{\omega}} := \sum_{\zeta_j \in \Upsilon_R^0} \omega_{\zeta_j} u_{\zeta_j} \tag{3.6}$$

Clearly, for any $\boldsymbol{\omega}$,

$$V_{\boldsymbol{\omega}} \geq \sum_{\zeta_j \in \Upsilon_R} \omega_{\zeta_j} u_{\zeta_j} = V_{\Upsilon_R^0,\boldsymbol{\omega}} + \sum_{\zeta_j \in \Upsilon_R \setminus \Upsilon_R^0} \omega_{\zeta_j} u_{\zeta_j} \tag{3.7}$$

Going to finite volumes, the same inequality holds with $\boldsymbol{\omega}$ replaced by $\boldsymbol{\omega}_\Lambda$.

We now follow [BK, GHK2, GK4]. Setting $K > 10\delta_-$, $\Lambda = \Lambda_L$, It follows from the lower bound in (1.3) that there exists a constant $c_{u,d} > 0$ such that

$$\overline{V}_{\Upsilon_R^0,\boldsymbol{\omega}_\Lambda}(x) := \frac{1}{(2RK)^d} \int_{\Lambda_{2RK}(0)} \mathrm{d}a\, V_{\boldsymbol{\omega}_\Lambda}(x-a) \geq \frac{c_{u,d}}{(2R)^d}\, Y_{\boldsymbol{\omega},\Lambda}\chi_\Lambda(x), \tag{3.8}$$

where

$$Y_{\boldsymbol{\omega},\Lambda} := \min_{\xi \in \widetilde{\Lambda}} \frac{1}{K^d} \sum_{\zeta \in \widetilde{\Lambda_{\frac{K}{3}}(\xi)}} \omega_\zeta\,. \tag{3.9}$$

It follows from standard estimates (e.g., [Y, Proposition 3.3.1]) that, with $\bar{\mu}$ the mean of the probability measure μ, we have, for some $A_\mu > 0$,

$$\mathbb{P}\left\{\frac{1}{K^d} \sum_{\zeta \in \widetilde{\Lambda_{\frac{K}{3}}(\xi)}} \omega_\zeta \leq \frac{\bar{\mu}}{2}\right\} \leq \mathrm{e}^{-A_\mu K^d}. \tag{3.10}$$

It follows from (3.8) and (3.10) that. with $c'_{u,d} = \frac{c_{u,d}}{2}$,

$$\mathbb{P}\left\{\overline{V}_{\boldsymbol{\omega}_\Lambda} > c'_{u,d}(2R)^{-d}\bar{\mu}\chi_\Lambda\right\} \geq 1 - L^d \mathrm{e}^{-A_\mu K^d}, \tag{3.11}$$

and thus, we have for the "free sites Hamiltonian" with

$$S = (D \setminus \Upsilon_R^0) \cap \Lambda, \tag{3.12}$$

(recall (3.3)-(3.4)), with probability $\geq 1 - L^d \mathrm{e}^{-A_\mu K^d}$, uniformly in $\mathbf{t}_S$,

$$\overline{H}_{\boldsymbol{\omega},\Lambda,\mathbf{t}_S} := -\Delta_\Lambda + V_{\boldsymbol{\omega}_\Lambda,\mathbf{t}_S} + \overline{V}_{\Upsilon_R^0,\boldsymbol{\omega}_\Lambda} \geq c'_{u,d}(2R)^{-d}\bar{\mu} \quad \text{on} \quad \mathrm{L}^2(\Lambda). \tag{3.13}$$

Thus, if $\varphi \in C_c^\infty(\Lambda)$ with $\|\varphi\| = 1$, we have

$$\begin{aligned}
\langle \varphi, H_{\boldsymbol{\omega},\Lambda,\mathbf{t}_S}\varphi\rangle_\Lambda &= \langle \varphi, \overline{H}_{\boldsymbol{\omega},\Lambda,\mathbf{t}_S}\varphi\rangle_\Lambda + \left\langle \varphi, \left(V_{\Upsilon^0_R,\boldsymbol{\omega}_\Lambda} - \overline{V}_{\Upsilon^0_R,\boldsymbol{\omega}_\Lambda}\right)\varphi\right\rangle_\Lambda \\
&\geq c'_{u,d}(2R)^{-d}\bar{\mu} + \left\langle \varphi, \left(V_{\Upsilon^0_R,\boldsymbol{\omega}_\Lambda} - \overline{V}_{\Upsilon^0_R,\boldsymbol{\omega}_\Lambda}\right)\varphi\right\rangle_{\mathbb{R}^d} \qquad (3.14)\\
&\geq \frac{c'_{u,d}}{(2R)^d}\bar{\mu} + \left\langle \varphi, V_{\Upsilon^0_R,\boldsymbol{\omega}_\Lambda}\varphi\right\rangle_{\mathbb{R}^d} - \frac{1}{(2RK)^d}\int_{\Lambda_{2RK}(0)} \mathrm{d}a\ \langle \varphi(\cdot + a), V_{\boldsymbol{\omega}_\Lambda}\varphi(\cdot + a)\rangle \\
&\geq \frac{c'_{u,d}}{(2R)^d}\bar{\mu} - \frac{1}{(2RK)^d}\int_{\Lambda_{2RK}(0)} \mathrm{d}a\ \left|\left\langle \varphi, V_{\Upsilon^0_R,\boldsymbol{\omega}_\Lambda}\varphi\right\rangle - \left\langle \varphi(\cdot + a), V_{\Upsilon^0_R,\boldsymbol{\omega}_\Lambda}\varphi(\cdot + a)\right\rangle\right| \\
&\geq \frac{c'_{u,d}}{(2R)^d}\bar{\mu} - 2c'_u KR\,\|\nabla_\Lambda\varphi\|_\Lambda \geq c'_{u,d}\,\bar{\mu} - 2c'_u KR\,\langle \varphi, H_{\boldsymbol{\omega},\Lambda}\varphi\rangle_\Lambda^{\frac{1}{2}},
\end{aligned}$$

where we used

$$\|\varphi(\cdot + a) - \varphi\|_{\mathbb{R}^d} = \left\|(\mathrm{e}^{a\cdot\nabla} - 1)\varphi\right\|_{\mathbb{R}^d} \leq |a|\,\|\nabla\varphi\|_{\mathbb{R}^d} = |a|\,\|\nabla_\Lambda\varphi\|_\Lambda\,. \qquad (3.15)$$

It follows that there is $\tilde{K}_{u,d} > 0$, such that for $K > \tilde{K}_{u,d}$ we have, uniformly in $\mathbf{t}_S$,

$$\langle \varphi, H_{\boldsymbol{\omega},\Lambda,\mathbf{t}_S}\varphi\rangle_\Lambda \geq c''_{u,d}\,\frac{\bar{\mu}^2}{R^{2d+2}K^2}. \qquad (3.16)$$

Since this holds for all $\varphi \in C_c^\infty(\Lambda)$ with $\|\varphi\| = 1$, we have, from (3.11), uniformly in $\mathbf{t}_S$,

$$H_{\boldsymbol{\omega},\Lambda,\mathbf{t}_S} \geq c''_{u,d}\,\frac{\bar{\mu}^2}{R^{2d+2}K^2} \quad \text{on } \mathrm{L}^2(\Lambda), \qquad (3.17)$$

with probability $\geq 1 - L^d\mathrm{e}^{-A_\mu K^d}$. Given $p > 0$, we take $K = \left(\frac{(p+1)d}{A_\mu}\log L\right)^{\frac{1}{d}}$ and get uniformly in $\mathbf{t}_S$,

$$\mathbb{P}\left\{H_{\boldsymbol{\omega},\Lambda_L,\mathbf{t}_S} \geq C_{u,\mu,d,p}R^{-(2d+2)}\left(\log L\right)^{-\frac{2}{d}}\right\} > 1 - L^{-pd}, \qquad (3.18)$$

for $L \geq \tilde{L}_{u,\mu,d,p}$, where $C_{u,\mu,d,p} > 0$ is an appropriate constant. □

Acknowledgements

It is the pleasure of the author to thank his co-authors on these recent results on localization, namely M. Aizenman, P. Hislop, A. Klein, S. Warzel, as well as P. Müller for interesting discussions on Delone sets.

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