Edge and Impurity Effects on Quantization of Hall Currents

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Abstract: We consider the edge Hall conductance and show it is invariant under perturbations located in a strip along the edge (decaying perturbations far from the edge are also allowed). This enables us to prove for the edge conductances a general sum rule relating currents due to the presence of two different media located respectively on the left and on the right half plane. As a particular interesting case we put forward a general quantization formula for the difference of edge Hall conductances in semi-infinite samples with and without a confining wall. It implies in particular that the edge Hall conductance takes its ideal quantized value under a gap condition for the bulk Hamiltonian, or under some localization properties for a random bulk Hamiltonian (provided one first regularizes the conductance; we shall discuss this regularization issue). Our quantization formula also shows that deviations from the ideal value occurs if a semi infinite distribution of impurity potentials is repulsive enough to produce current-carrying surface states on its boundary.

1. Introduction

There has been recently some renewed interest in detailed analysis of edge states occuring in semi-infinite quantum Hall systems, which play a basic role in the analysis of the quantum Hall effect (for a general reference to the QHE, see e.g. [PG]). Such edge states have been proved to carry currents at least in weak disorder regimes [DBP,FGW1,FGW2,FM1,FM2,CHS]. These discussions need to be completed by an analysis of the quantization properties of these currents and of the effect of various types of perturbations, like edge imperfections or random impurities, on these quantized values. The role of edge states in quantization of

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Hall conductance has been widely discussed since the pioneering work of B.I. Halperin [H] (see e.g. [HT, MDS, B, Th, CFGP] and references therein). It has been shown recently in [SBKR, KRSB, EG, Ma] that for discrete Hamiltonians with a magnetic field and under a *gap condition* of the bulk Hamiltonian the edge theory and the bulk theory can be reconcilied and the edge conductance as defined in Definition 1 equals the bulk conductance as given by Kubo's formula provided the Fermi energy lies in such a gap[1]. Let us recall that the bulk conductance has received an interpretation both as a Chern number [BESB] and as a topological invariant [Ku, AS2], thus providing an explanation for both quantization and robustness of Hall conductance. In the ergodic case and under a *gap condition* the edge conductance can also be expressed as a Fredholm index [SBKR, KRSB, KSB]. However, as compared to the bulk theory (e.g. [Be, Ku, AS2, BESB, AG, ES, BGKS]) some of the main arguments of the edge theory for the quantum Hall effect have not been given yet a rigourous mathematical status, efficient enough quantitatively to deal with the questions mentioned above. One goal of this paper is to compute the edge conductance in a simple way, *independently of a gap assumption*, and to study its stability under perturbations. We note that the exact quantization is obtained here without any covariant structure of the Hamiltonians.

One of our main results is a general sum rule linking the conductances of the same system with and without the confining edge (Corollary 3). It is obtained as a particular case of Theorem 2 which deals with general left and right media. We shall provide two models with random impurities for which the edge conductance either vanishes or keeps its ideal quantized value N, when the Fermi energy lies between the N^{th} and $(N+1)^{\text{th}}$ Landau levels. The first model is the one of Nakamura and Bellissard [NB] that we adapt to the edge geometry. We recover in a simple way their result but from the "edge" point of view, i.e. we prove the vanishing of the edge conductance. As a result this implies the existence of a persistent current carrying states due to the impurity potential alone and living near the boundary of the disordered region; these currents are shown to be quantized as well. The second model is of Anderson type, and we investigate the edge conductance in the regime of localized states, in which case a regularization of the edge conductance is required[2]. We shall discuss this regularization issue, and show that under a suitable condition of localization the regularized edge conductance keeps its ideal quantized value N.

2. Statements of the general results

Throughout this paper $\mathbf{1}_X = \mathbf{1}_{(x,y)}$ will denote the characteristic function of a unit cube centered $X = (x,y) \in \mathbb{Z}^2$. If A is a subset of $\mathbb{R}^2$, then $\mathbf{1}_A$ will denote the characteristic function of this set. Moreover $\mathbf{1}_-$ and $\mathbf{1}_+$ will stand, respectively, for $\mathbf{1}_{x\leq 0}$ and $\mathbf{1}_{x\geq 0}$.

We consider an electron confined to the 2-dimensional plane composed of two complementary semi-infinite regions supporting potentials V_1 and V_2 respectively, and under the influence of a constant magnetic field B orthogonal to the

[1] While writing the revised version of this paper, we heard of the recent work of A. Elgart, G.M. Graf, J. Schenker [EGS] concerning the equality of the bulk and edge conductances in a *mobility gap*, namely in a region where one has localized states.

[2] The regularization issue is also treated in [EGS] (see Footnote 1).

sample. If V_1, V_2 are two potentials in the Kato class [CFKS] the Hamiltonian of the system is given, in suitable units and Landau gauge, by

$$H(V_1,V_2) := H_L + V_1\mathbf{1}_- + V_2\mathbf{1}_+ \,, \tag{2.1}$$

a self-adjoint operator acting on $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x\mathrm{d}y)$, where $H_L = p_x^2 + (p_y - Bx)^2$ is the free Landau Hamiltonian. The spectrum of $H_L = H(0,0)$ consists in the well-known Landau levels $B_N = (2N-1)B$, $N \geq 1$ (with the convention $B_0 = -\infty$). For technical reasons it is convenient to assume the following control on the growth at infinity of V_1, V_2: for some uniform constants $C, p > 0$,

$$\|\mathbf{1}_{(x,y)}V_1\|_\infty \leq C\langle x\rangle^p, \text{ if } x \leq 0, \quad \text{and} \quad \|\mathbf{1}_{(x,y)}V_2\|_\infty \leq C\langle x\rangle^p, \text{ if } x \geq 0. \tag{2.2}$$

For simplicity we further assume that the potentials V_1, V_2 are bounded from below, so that $H(V_1,V_2)$ is a bounded from below self-adjoint operator.

We shall say that V_1, resp. V_2, is a (left), resp. (right), *confining potential* with respect to the interval $I = [a,b] \subset \mathbb{R}$ if in addition to the previous conditions the following holds: there exists $R > 0$, s.t.

$$\forall x \leq -R, \forall y \in \mathbb{R},\ V_1(x,y) > b, \quad \text{resp.} \quad \forall x \geq R, \forall y \in \mathbb{R},\ V_2(x,y) > b. \tag{2.3}$$

The "hard wall" case where V_1 is infinite and $H = H_L + V_2$ acts on $\mathrm{L}^2(\mathbb{R}^+ \times \mathbb{R}, \mathrm{d}x\mathrm{d}y)$ with Dirichlet boundary condition at $x = 0$ can also be considered, and our results still hold.

As typical examples for $H(V_1,V_2)$ one may think of the right potential V_2 as an impurity potential and of the left potential V_1 as either a wall, confining the electron to the right half plane and generating an edge current, or an empty region ($V_1 = 0$), in which case the issue is to determine whether or not V_2 is strong enough to create edge currents by itself. Another example is the strip geometry, where both V_1 and V_2 are confining.

Following [SBKR, KRSB, EG, Ma] we adopt the following definition of an edge conductance. Define a "switch" function as a smooth real valued *increasing* function equal to 1 (resp. 0) at the right (resp. left) of some bounded interval; then

Definition 1. *Let* $\mathcal{X} \in \mathcal{C}^\infty(\mathbb{R}^2)$ *be a x-translation invariant switch function with* $\mathrm{supp}\mathcal{X}' \subset \mathbb{R} \times [-\frac{1}{4}, \frac{1}{4}]$*, and let* $-g \in \mathcal{C}^\infty(\mathbb{R})$ *be switch a function with* $\mathrm{supp}\, g' \subset I = [a,b]$ *a compact interval. The* edge conductance[3] *of* $H = H(V_1,V_2)$ *in the interval* I*, is defined as*

$$\sigma_e(g,H) \equiv \sigma_e(g,V_1,V_2) := -\mathrm{tr}(g'(H(V_1,V_2)) i[H(V_1,V_2), \mathcal{X}]) \tag{2.4}$$
$$= -\mathrm{tr}(g'(H(V_1,V_2)) i[H_L, \mathcal{X}]) \tag{2.5}$$

whenever the trace is finite (we shall use both expressions $\sigma_e(g,V_1,V_2)$ *and* $\sigma_e(g,H(V_1,V_2))$*).*

[3] As suggested to us by one referee, one could also call $\sigma_e(g,V_1,V_2)$ the *interface* conductance between potentials V_1 and V_2. Concerning the physical interpretation of this quantity, see comments below.

Remark 1. Since $[H_L, \mathcal{X}]$ is relatively $H(V_1,V_2)$ bounded with relative bound 0, the operator $g'(H(V_1,V_2))i[H_L,\mathcal{X}]$ readily extends to a bounded operator on $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x\mathrm{d}y)$. The only issue is thus the finiteness of the trace. In the strip geometry the trace is always well defined, and is actually zero (Corollary 2). In the one wall case, say V_1 is left confining, the situation is very different: if I is in a gap of $H(0,V_2)$ then $g'(H(V_1,V_2))i[H_L,\mathcal{X}]$ will be shown to be trace class (Corollary 4); but without the gap condition the situation is more delicate, and a regularized version of (2.4) is needed; we shall discuss this point in Section 7.

Remark 2. In the situations of interest $\sigma_e(g,V_1,V_2)$ will turn out to be independant of the particular shape of the switch function $\mathcal{X}$ and also of the switch function g, provided $\mathrm{supp} g'$ does not contain any Landau level.

In practice, in this paper, we shall mainly focus on the following two simple situations: (i) the potential V_1 plays the role of a potential barrier (soft wall), (ii) $V_1 = 0$ in which case we investigate the influence of the sole impurities potential V_2. So in both situations we are interested in the possible existence of *edge* currents. In cases (i) and (ii) $\sigma_e(g,H)$ can be understood in physical terms as follows. Take g to be piecewise linear so that $g'=0$ outside $[a,b]$ and $-g'(H) = E_H(I)/|b-a|$, with $E_H(I)$ the spectral projection of H on $I=[a,b]$. The edge conductance $\sigma_e(g,H)$ is then seen as the ratio $J(I)/|b-a|$, where $J(I) = \mathrm{tr}(E_H(I)i[H,\mathcal{X}])$ is the total current through the surface $y=0$ induced by states with energy support contained in I. We note that in case (i), i.e. the one wall case, $J(I)$ can be interpreted as the total current flowing in a strip whose edges are at different chemical potential $E_- = a$ and $E_+ = b$, as discussed in [SBKR]; this assumes that edges are well-separated to prevent effective tunneling between both edges, so that such a strip can in turn be represented by two copies of one edge (half-plane) Hamiltonian with edge currents flowing in opposite directions (for other discussions about this picture, see e.g. [H, HT, MDS, Th]).

Our first result is the

Theorem 1. *Let $H = H(V_1,V_2)$ be as in (2.1), and let W be a bounded potential supported in a strip $[L_1,L_2]\times\mathbb{R}$, with $-\infty < L_1 < L_2 < +\infty$. Then the operator $(g'(H+W) - g'(H))i[H_L,\mathcal{X}]$ is trace class, and*

$$\mathrm{tr}((g'(H+W) - g'(H))i[H_L,\mathcal{X}]) = 0. \tag{2.6}$$

As a consequence:
(i) $\sigma_e(g, H_L + W) = 0$.
(ii) Assume V_1 is a y-invariant potential, i.e. $V_1(x,y) = V_1(x)$, that is left confining with respect to $I \supset \mathrm{supp} g'$. If $I \subset]B_N, B_{N+1}[$, for some $N \geq 0$, then

$$\sigma_e(g, H_L + V_1 + W) = N. \tag{2.7}$$

Remark 3. The hypotheses on the strip geometry of W in Theorem 1 can be relaxed to some extent. It follows from the proof (see bound (4.16)) that a fast enough decaying potential W in the x-direction works as well; for instance $\sup_{x_1}\langle x_1\rangle^{k_1}\|W\mathbf{1}_{(x_1,y_1)}\| < C\langle y_1\rangle^{k_2}$ is fine provided k_1 is large enough (but k_2 can be anything).

That $\sigma_e(g, V_1, 0) = N$, with V_1 a y-invariant left confining potential, is an easy consequence of the spectral properties of $H(V_1, 0)$ (Proposition 1). In this case the current is carried by edge states which are localized within a few cyclotron radius from the edge [DBP, FGW1, FM1, FM2, CHS]. Property (2.6) implies that a bounded perturbation localized in a strip will not affect the total current, but only, possibly, the geometry of its flow. One can imagine in particular that a strongly repulsive W will move all the current carrying states at the right of the strip supporting W. On the other hand, if the potential is small, edge states will survive near $x = 0$ and will still propagate along the wall V_1.

As a first corollary of Theorem 1, we note that to a large extent edge conductances do not depend on the confining potential V_1 so that irregular confining boundaries are allowed.

Corollary 1. *Let $V_1^{(i)}$, $i = 1, 2$, be two left confining potentials with respect to $[a, b] \supset \mathrm{supp} g'$, and $H_i := H(V_1^{(i)}, V_2)$. If $V_1^{(1)} - V_1^{(2)}$ is supported in a strip, then $(g'(H_1) - g'(H_2))[H_L, \mathcal{X}]$ is trace class with trace zero. In particular if one conductance is finite, so is the second one, and $\sigma_e(g, V_1^{(1)}, V_2) = \sigma_e(g, V_1^{(2)}, V_2)$.*

Remark 4. Notice that we do not assume that these confining potentials are y-invariant. So if $V_1^{(1)}$ is a y-invariant left confining potential, then any distortion $V_1^{(2)}$ of the boundary that is supported in a strip or, according to Remark 3, that decays fast enough as $x \to -\infty$, will leave the edge conductance invariant, i.e. $\sigma_e(g, V_1^{(2)}, 0) = N$. However the nature of the spectrum of H_1 may change. For instance the proof in [FGW2] of the absolutely continuity of the spectrum of $H(V_1, 0)$ requires some smoothness of the boundary of the support of V_1.

Our second corollary of Theorem 1 investigates the case of the strip geometry.

Corollary 2. *Let $\tilde{V}_0(x, y)$ be a left* and *right confining potential, s.t. $\tilde{V}_0(x, y) \geq v_0 > B_{N+1}$ if $|x| > R$, and $\tilde{V}_0(x, y) = 0$ if $|x| \leq R$. Then for any electrostatic bounded potential $U(x, y)$ contained in $|x| \leq R$, and any g, with* $\mathrm{supp} g' \subset]-\infty, B_{N+1}[$, *one has*

$$\sigma_e(g, H_L + \tilde{V}_0 + U) = 0\,. \tag{2.8}$$

Remark 5. Eq. (2.8) states that there is no total current flowing in a strip at equilibrium, even in presence of an electrostatic field. When U is zero, this result also follows from the spectral analysis of H_0 (see e.g. [CHS]) showing that both edges carry opposite currents (if any). Impurities and electrostatic potential just have the effect of modifying the geometry of the flow of edge currents, but in such a way that they always compensate and sum up to zero.

So far we only considered perturbations located in a strip of the type $[L_1, L_2] \times \mathbb{R}$. But what happens when the right boundary L_2 of the strip potential is taken to infinity? It is easy to check that Theorem 1 does not extend as it stands. Consider $W_\ell = v_0 \mathbf{1}_{[0,\ell]}(x)$ and $W_\infty = v_0 \mathbf{1}_{[0,\infty[}(x)$, with the constant $v_0 \geq B_{N+1}$, then Theorem 1 yields $\sigma_e(g, 0, W_\ell) = 0$ for all $\ell > 0$ while $\sigma_e(g, 0, W_\infty) = -N$. In other terms, adding a potential W that does not decay at infinity may dramatically perturbe the existence of edge currents.

However from Theorem 1 we get that for any bounded potential W supported on a strip $[L_1, L_2] \times \mathbb{R}$, one has

$$\sigma_e(g, V_1, W) - \sigma_e(g, 0, W) = \sigma_e(g, V_1, 0) - \sigma_e(g, 0, 0) = N. \tag{2.9}$$

Although Theorem 1 does not extend in the limit $L_2 \to \infty$, it turns out that the difference rule (2.9) does. We shall give a rigorous content of this fact in Corollary 3, which is a particular case of our second theorem.

Theorem 2. *Let g be s.t.* $\operatorname{supp} g' \subset]B_N, B_{N+1}[$ *for some $N \geq 0$. Then the operator $\{g'(H(V_1,V_2)) - g'(H(V_1,0)) - g'(H(0,V_2))\} i[H_L, \mathcal{X}]$ is trace class, and*

$$\operatorname{tr}(\{g'(H(V_1,V_2)) - g'(H(V_1,0)) - g'(H(0,V_2))\} i[H_L, \mathcal{X}]) = 0. \tag{2.10}$$

In a similar way, let V_0 be as in (2.3) *a confining potential with respect to* $\operatorname{supp} g'$ *(left or right depending on where V_0 is supported*[4]*), then the operator $\{g'(H(V_1,V_2)) - g'(H(V_1,V_0)) - g'(H(V_0,V_2))\} i[H_L, \mathcal{X}]$ is trace class, and*

$$\operatorname{tr}(\{g'(H(V_1,V_2)) - g'(H(V_1,V_0)) - g'(H(V_0,V_2))\} i[H_L, \mathcal{X}]) = 0. \tag{2.11}$$

In particular, if traces are separately finite then

$$\sigma_e(g, V_1, V_2) = \sigma_e(g, V_1, 0) + \sigma_e(g, 0, V_2) \tag{2.12}$$
$$= \sigma_e(g, V_1, V_0) + \sigma_e(g, V_0, V_2). \tag{2.13}$$

Remark 6. (i) If $\operatorname{supp} g'$ contains one (or more) Landau levels, then the trace in (2.10) is no longer zero, but is equal to $\operatorname{tr}(g'(H_L) i[H_L, \mathcal{X}]) = -\sigma_e(g, H_L) \neq 0$.
(ii) If V_0 is not confining, then the operator in (2.11) should be replaced by $\{g'(H(V_1,V_2)) - g'(H(V_1,V_0)) - g'(H(V_0,V_2)) + g'(H(V_0,V_0))\} i[H_L, \mathcal{X}]$.
(iii) If V_1 is confining or if $\operatorname{supp} g'$ lies in a gap of $H(V_1, V_1)$ (so that in both cases $\sigma_e(g, V_1, V_1) = 0$), then it follows from (2.12) that $\sigma_e(g, V_1, 0) = -\sigma_e(g, 0, V_1)$.

As an immediate consequence of Theorem 2 we get a quantization rule for the difference of the edge conductances with and without a confining potential V_1, that shows that they are simultaneously quantized.

Corollary 3. *Let g be s.t.* $\operatorname{supp} g' \subset]B_N, B_{N+1}[$, *for some $N \geq 0$. Let V_1 be a y-invariant left confining potential with respect to* $\operatorname{supp} g'$ *or a perturbation of such a V_1 as in Corollary 1. Then the operator $\{g'(H(V_1,V_2)) - g'(H(0,V_2))\} i[H_L, \mathcal{X}]$ is trace class and*

$$-\operatorname{tr}(\{g'(H(V_1,V_2)) - g'(H(0,V_2))\} i[H_L, \mathcal{X}]) = N. \tag{2.14}$$

In particular, if either $\sigma_e(g, V_1, V_2)$ or $\sigma_e(g, 0, V_2)$ is finite, then both are finite, and

$$\sigma_e(g, V_1, V_2) - \sigma_e(g, 0, V_2) = N. \tag{2.15}$$

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[4] Strictly speaking if V_0 is left confining, then $V_0^*(x,y) = V_0(-x,y)$ is right confining. With some abuse of notations we still write V_0 instead of V_0^* if we consider the right confining potential.

Note that $\sigma_e(g,0,V_2) \neq 0$ would imply the existence of current carrying states due to the sole impurity potential. Since Corollary 3 would yield $\sigma_e(g,V_1,V_2) \neq N$, we see that such "edge currents without edges" are responsible for the deviation of the Hall conductance from its ideal value N. An example of this phenomenon is provided by the model of S. Nakamura and J. Bellissard in [NB] that we shall revisit in section 6.

On the other hand, if the potential V_2 is not strong enough to close the Landau gaps and if the Fermi level falls into a gap of $H(0,V_2)$, then obviously $\sigma_e(g,0,V_2) = 0$, and Corollary 3 immediately gives the exact quantized value of the edge conductance. In particular we recover the fact that the conductance remains constant if one increases the coupling constant while keeping the Fermi level in a gap [AS2, BESB, ES]. We thus have the

Corollary 4. *Let g and V_1 as in Corollary 3, $N \geq 0$. If* $\mathrm{supp} g'$ *belongs to a gap of $H(0,V_2)$, then $\sigma_e(g,V_1,V_2) = N$. As a consequence, let $\lambda^* > 0$ s.t. $\|\lambda^* V_2\| < B$ and g s.t.* $\mathrm{supp} g' \subset]B_N + \|\lambda^* V_2\|, B_{N+1} - \|\lambda^* V_2\|[$, *then*

$$\forall \lambda \in [0,\lambda^*],\ \sigma_e(g,V_1,\lambda V_2) = N, \tag{2.16}$$

If now $\mathrm{supp} g'$ is no longer included in a gap of $H(0,V_2)$, but in a region of localization, then one expects a regularized version of $\sigma_e(g,0,V_2)$ to be still zero (the aim of the regularization is to restore the trace class property of $g'(H(0,V_2))i[H_L,\mathcal{X}]$ that fails in a region of localization). In this case the analog of Corollary 4 holds for the regularized conductances, i.e. $\sigma_e^{\mathrm{reg}}(g,V_1,\lambda V_2) = N$, thus recovering from the "edge point of view" the bulk picture [BESB, AG]. This regularization issue is the content of Section 7.

Remark 7. As a by-product we recover a posteriori the equality "bulk-edge" of the conductances for in the context of Corollary 4 the bulk conductance is also known to be equal to N [BESB, AS2].

The plan of the paper is as follows. In Section 3 we recall by direct computation that the results stated in (2.7) hold in absence of impurities (free case). In Section 4 we prove Theorem 1; we first show a simple invariance property for $\sigma_e(g,H)$ under a perturbation by a compactly supported potential; this invariance property is extended to potentials supported in a strip (or more generally decaying potential in the x direction) by Combes-Thomas arguments together with Helffer-Sjöstrand functional calculus. In Section 5 we prove Theorem 2 on the account of Theorem 1. In Section 6 we revisit the model of Nakamura and Bellissard [NB] and get an example of a zero edge conductance due to a strongly repulsive potential. Section 7 is devoted to the case where $\mathrm{supp} g'$ does not lie anymore in a gap, but in a region of localized states. We introduce a regularization and recover the sum rule of Corollary 3 for the regularized edge conductances together with the analog of Corollary 4 in mobility gaps. Appendices A and B contain tools and estimates we shall make use of throughout the paper.

3. Edge conductance of the unperturbed operator

The following result is well-known. For the sake of completeness we shall provide a short proof of it.

Proposition 1. *Let* $I=[a,b]\subset]B_N,B_{N+1}[$ *be such that* $I\supset \mathrm{supp} g'$. *We have*

$$\sigma_e(g,0,0)=0. \tag{3.1}$$

Assume that V_1 *is a left confining potential with respect to* I. *Then the operator* $g'(H(V_1,0))i[H_L,\mathcal{X}]$ *is trace class. If in addition* V_1 *is* y*-invariant, then one has*

$$\sigma_e(g,V_1,0)=N\,. \tag{3.2}$$

Remark 8. In the next section, we will show that (3.2) also holds if the confining potential V_1 has imperfections (i.e. V_1 may depend on y as well). See Remark 4. Moreover, it actually follows from the proof that one can add to the confining potential V_1 any bulk mean electrostatic field V_2 depending only on x and vanishing at $+\infty$: one still has $\sigma_e(g,V_1,V_2)=N$.

Remark 9. The same proof with $V_1^*(x):=V_1(-x)$ gives $\sigma_e(g,0,V_1^*)=-N$.

Proof. That $\sigma_e(g,0,0)=0$ is immediate since $\sigma(H_L)\cap I=\emptyset$. We turn to the free edge Hamiltonian $H_0:=H(V_1,0)=H_L+V_1\mathbf{1}_-$. That $g'(H_0)i[H_L,\mathcal{X}]$ is trace class follows from the arguments developped in this paper (more precisely those of Sections 4 and 5), and the proof is sketched in Appendix B, Lemma 5.

We now compute the trace itself. Due to the invariance by translation in the y direction, we perform a partial Fourier transform in the y variable and write,

$$H_0\simeq\int_{\mathbb{R}}^{\oplus}H_0(k)\mathrm{d}k,\quad H_0(k)=p_x^2+(k-Bx)^2+V_1(x)\mathbf{1}_-\,. \tag{3.3}$$

We refer to [DBP, FGW1, CHS] for details on this operator. Eigenfunctions of the one-dimensional Hamiltonian $H_0(k)$, $k\in\mathbb{R}$, will be denoted $\xi_{n,k}(x)$, $n=1,2,\cdots$, with eigenvalue $\omega_n(k)$ ordered increasingly. Assumption on V_1 at $\pm\infty$ implies that $\omega_n(+\infty)=\lim_{k\to+\infty}\omega_n(k)=(2n+1)B$ and $\omega_n(-\infty)=\lim_{k\to-\infty}\omega_n(k)>b$. It follows that $g(\omega_n(+\infty))=1$ if $n\le N$ and zero if $n>N$, while $g(\omega_n(-\infty))$ is always zero. Generalized eigenfunctions of H_0 then read $\varphi_{n,k}(x,y)=\mathrm{e}^{iky}\xi_{n,k}(x)$, $n=1,2,\cdots$ and $k\in\mathbb{R}$. Note that from the Feynman-Hellman formula,

$$\omega_n'(k)=2\langle\xi_{n,k},(k-Bx)\xi_{n,k}\rangle\,. \tag{3.4}$$

It follows that (with some abuse of notation we denote again by $\mathcal{X}(y)$ the one-dimensional function equal to $\mathcal{X}(x,y)$ for all $x\in\mathbb{R}$)

$$\sigma_e(g,H_0)=-2\sum_{n\ge1}\int_{\mathbb{R}}g'(\omega_n(k))\langle\varphi_{n,k},(k-Bx)\mathcal{X}'(y)\varphi_{n,k}\rangle\mathrm{d}k \tag{3.5}$$

$$=\sum_{n\ge1}\left(g(\omega_n(+\infty))-g(\omega_n(-\infty))\right)=\sum_{n\ge1}g(\omega_n(+\infty))=N, \tag{3.6}$$

where we used in (3.5) that $\int_{\mathbb{R}}\mathcal{X}'(y)\mathrm{d}y=\mathcal{X}(1)-\mathcal{X}(0)=1$. □

4. Perturbation by a strip potential

The aim of this section is to prove Theorem 1. But, given Theorem 1, we first show how to get Corollary 2: by Theorem 1, $\sigma_e(g, H_L + \tilde{V}_0) = \sigma_e(g, H_L + \tilde{V}_0 + v_0\mathbf{1}_{[-R,R]}) = 0$ (since $g'(H_L + \tilde{V}_0 + v_0\mathbf{1}_{[-R,R]}) = 0$); applying a second time Theorem 1 gives $\sigma_e(g, H_L + \tilde{V}_0 + U) = \sigma_e(g, H_L + \tilde{V}_0) = 0$.

To prove Theorem 1, we proceed in two steps. First we show that edge conductances are invariant under a perturbation by a bounded and compactly supported potential (Lemma 1); then we extend the result to strip potentials (or decaying potential in the x-direction as pointed in Remark 3).

Lemma 1. *Let $\Lambda \subset \mathbb{R}^2$ be compact and W a bounded potential supported on Λ. Let H be as in* (2.1). *Then $(g'(H+W) - g'(H))i[H_L, \mathcal{X}] \in \mathcal{T}_1$ and*

$$\mathrm{tr}((g'(H+W) - g'(H))i[H_L, \mathcal{X}]) = 0\,. \tag{4.1}$$

Proof. To compare the operators $g'(H+W)$ and $g'(H)$, we shall make use of the Helffer-Sjöstrand formula [HeSj,HuSi]. Let $\tilde{g}_n$ be a quasi-analytic extension of g or order $n \geq 3$ (see Appendix A). Then, writing $R_\Lambda^2(z) = (H + W - z)^{-2}$ and $R^2(z) = (H - z)^{-2}$, (A.2) reads

$$g'(H+W) - g'(H) = -\frac{1}{\pi}\int \bar{\partial}\tilde{g}_n(u+iv)(R_\Lambda^2(z) - R^2(z))\mathrm{d}u\mathrm{d}v, \quad z = u+iv. \tag{4.2}$$

Note that $\mathrm{Im}z \neq 0$. For further reference recall the second order resolvent: if H_1 and $H_2 = H_1 + W$ are two self-adjoint operators, $R_i = (H_i - z)^{-1}$, then

$$R_2^2 - R_1^2 = -R_2R_1WR_2 - R_1WR_2R_1. \tag{4.3}$$

Since W has a compact support, both operators $R_\Lambda RW$ and RWR_Λ are in $\mathcal{T}_1$ according to Lemma 4. Moreover both $R_\Lambda[H_\Lambda, \mathcal{X}]$ and $R[H, \mathcal{X}]$ extend to bounded operators. As a consequence, using (4.3),

$$\mathrm{tr}\left((R_\Lambda^2 - R^2)[H, \mathcal{X}]\right) = -\mathrm{tr}\left(R_\Lambda RWR_\Lambda[H, \mathcal{X}]\right) - \mathrm{tr}\left(RWR_\Lambda R[H, \mathcal{X}]\right), \tag{4.4}$$

each trace being finite for operators are actually trace class, and the first statement of the Lemma follows. Suppose now we have shown that

$$\mathrm{tr}\left(R_\Lambda RWR_\Lambda[H, \mathcal{X}]\right) = \mathrm{tr}\left(RWR_\Lambda[H, \mathcal{X}]R_\Lambda\right). \tag{4.5}$$

Since $RWR_\Lambda \in \mathcal{T}_1$ and $R[H, \mathcal{X}]$ is bounded, we also have

$$\mathrm{tr}\left(RWR_\Lambda R[H, \mathcal{X}]\right) = \mathrm{tr}\left(R[H, \mathcal{X}]RWR_\Lambda\right). \tag{4.6}$$

Thus, taking advantage of $R[H, \mathcal{X}]R = [R, \mathcal{X}]$, (4.4) reduces to

$$\mathrm{tr}\left((R_\Lambda^2 - R^2)[H, \mathcal{X}]\right) = \mathrm{tr}\left(RWR_\Lambda\mathcal{X}\right) - \mathrm{tr}\left(\mathcal{X}RWR_\Lambda\right) = 0. \tag{4.7}$$

Since by Lemma 4 the integral in (4.2) is absolutely convergent in $\mathcal{T}_1$, we can pass the trace inside the integral and get (4.1).

We come back to (4.5). If $M < \inf\sigma(H_\Lambda)$, then $R_\Lambda(M)^{1/2}R(z)W$ can be shown to be trace class. Indeed, by the resolvent identity

$$R_\Lambda(M)^{\frac{1}{2}}R(z)W = R_\Lambda(M)^{\frac{3}{2}}W + R_\Lambda(M)^{\frac{3}{2}}(z - M - W)R(z)W; \tag{4.8}$$

now, since W is compactly supported, $R_\Lambda(M)^{3/2}W \in \mathcal{T}_1$ (e.g. [Si] or [GK2, Lemma A.4]) and the operators $R_\Lambda(M)^{\frac{3}{2}}WR(z)$ and $(z-M)R_\Lambda(M)^{3/2}R(z)W$ belong to $\mathcal{T}_1$ by Lemma 4. Thus

$$\begin{aligned}
&\operatorname{tr}\left(R_\Lambda RWR_\Lambda[H,\mathcal{X}]\right)\\
&= \operatorname{tr}\left(R_\Lambda(z)(H_\Lambda - M)R_\Lambda(M)^{\frac{1}{2}}R_\Lambda(M)^{\frac{1}{2}}R(z)WR_\Lambda(z)[H,\mathcal{X}]\right)\\
&= \operatorname{tr}\left(R_\Lambda(M)^{\frac{1}{2}}R(z)WR_\Lambda(z)[H,\mathcal{X}]R_\Lambda(z)(H_\Lambda - M)R_\Lambda(M)^{\frac{1}{2}}\right)\\
&= \operatorname{tr}\left(R(z)WR_\Lambda(z)[H,\mathcal{X}]R_\Lambda(z)(H_\Lambda - M)R_\Lambda(M)\right) = \operatorname{tr}\left(RWR_\Lambda[H,\mathcal{X}]R_\Lambda\right).
\end{aligned}$$

We applied the cyclicity property of the trace twice: the first time thanks to $R_\Lambda(M)^{1/2}R(z)W \in \mathcal{T}_1$, and the second time because $RWR_\Lambda \in \mathcal{T}_1$ according to Lemma 4. □

Proof (Proof of Theorem 1). The potential W is now supported on a strip $[L_1, L_2]\times\mathbb{R}$. We decompose W in the y direction and write, with obvious notations, $W = W_{>R} + W_{\leq R}$, for $R>0$. It follows from Lemma 1 that $(g'(H+W) - g'(H+W_{>R}))i[H,\mathcal{X}] \in \mathcal{T}_1$ and its trace is zero, for the difference between $H+W$ and $H+W_{>R}$ is the compactly supported potential $W_{\leq R}$. It thus remains to show that $\|(g'(H+W_{>R}) - g'(H))i[H,\mathcal{X}]\|_1$ goes to zero as R tends to infinity.

As in Lemma 1, we use the Helffer-Sjöstrand formula (A.2) together with the second order resolvent equation (4.3). We denote respectively by R and $R_{>R}$ the resolvents of H and $H+W_{>R}$. One has

$$\begin{aligned}
&\|(g'(H+W_{>R}) - g'(H))i[H,\mathcal{X}]\|_1 &(4.9)\\
&\leq \frac{1}{\pi}\iint |\bar\partial\tilde g(u+iv)|\, \left\|(R_{>R}(u+iv)^2 - R(u+iv)^2)i[H,\mathcal{X}]\right\|_1 \mathrm{d}u\mathrm{d}v. &(4.10)
\end{aligned}$$

Write, with $z = u+iv$,

$$-(R^2(z) - R^2_{>R}(z)) = R(z)R_{>R}(z)W_{>R}R(z) + R_{>R}(z)W_{>R}R(z)R_{>R}(z), \tag{4.11}$$

Let $\tilde{\mathcal{X}}$ be a smooth function such that $\tilde{\mathcal{X}} = 1$ on $\mathbb{R}\times[-\frac{1}{4},\frac{1}{4}]$ and $\tilde{\mathcal{X}} = 0$ outside $\mathbb{R}\times[-\frac{1}{2},\frac{1}{2}]$ (in particular $\tilde{\mathcal{X}} = 1$ on the support of $\mathcal{X}'$). So $[H,\mathcal{X}] = [H,\mathcal{X}]\tilde{\mathcal{X}}$. We divide $\tilde{\mathcal{X}}$ into cubes by writing $\tilde{\mathcal{X}} = \sum_{x_2\in\mathbb{Z}} \mathbf{1}_{(x_2,0)}$, with $\mathbf{1}_{(x_2,0)}$ being smooth functions. Let us also write

$$\mathbf{1}_{[L_1,L_2]\times[-R,R]^c} = \sum_{x_1\in\mathbb{Z}\cap[L_1,L_2]}\ \sum_{y_1\in\mathbb{Z},\,|y_1|>R} \mathbf{1}_{(x_1,y_1)}. \tag{4.12}$$

For any $(x_1,y_1)\in\mathbb{Z}^2\cap([L_1,L_2]\times[-R,R]^c)$, we have

$$\begin{aligned}
&\|RR_{>R}\mathbf{1}_{(x_1,y_1)}WR[H,\mathcal{X}]\tilde{\mathcal{X}}\|_1 &(4.13)\\
&\leq \sum_{x_2\in\mathbb{Z}} \|RR_{>R}\mathbf{1}_{(x_1,y_1)}\|_1 \|W\mathbf{1}_{(x_1,y_1)}\| \|\mathbf{1}_{(x_1,y_1)}R[H,\mathcal{X}]\mathbf{1}_{(x_2,0)}\| &(4.14)\\
&\leq \frac{C}{\eta}\|RR_{>R}\mathbf{1}_{(x_1,y_1)}\|_1\|W\mathbf{1}_{(x_1,y_1)}\| \sum_{x_2\in\mathbb{Z}} \langle x_2\rangle\, \mathrm{e}^{-c\eta(|x_1-x_2|+|y_1|)}, &(4.15)
\end{aligned}$$

where to get the last inequality we used Lemma 3, Eq. A.8, together with the Combes-Thomas estimate (A.4) and $\eta = \mathrm{dist}(z, \sigma(H))$. Summing over x_2, we get from (4.15) and Lemma 4,

$$\|RR_{>R}\mathbf{1}_{(x_1,y_1)}V_\Lambda R[H,\mathcal{X}]\|_1 \leq \frac{C}{\eta^\kappa}\|W\mathbf{1}_{(x_1,y_1)}\|\langle x_1\rangle \mathrm{e}^{-c\eta|y_1|}, \tag{4.16}$$

where κ stands for a positive integer (its value will vary, like the one of the constant C). It remains to sum over $x_1 \in [L_1, L_2]$ and $|y_1| \geq R$. It yields

$$\|RR_{>R}W_{>R}R[H,\mathcal{X}]\|_1 \leq \frac{C(L_2-L_1)\|W\|_\infty}{\eta^\kappa}\mathrm{e}^{-c\eta R}. \tag{4.17}$$

We turn to the second term coming from the decomposition of $R_\Lambda^2(z) - R_{>R}^2(z)$ in (4.11). As above we have to control

$$\|R_{>R}\mathbf{1}_{(x_1,y_1)}WRR_{>R}[H,\mathcal{X}]\mathbf{1}_{(x_2,0)}\|_1. \tag{4.18}$$

The trace class property will follow from the part $R_{>R}\mathbf{1}_{(x_1,y_1)}WR$, but we also need the term $\mathbf{1}_{(x_1,y_1)}$ to extract the required decay in y_1. We thus first pass a smooth version of $\mathbf{1}_{(x_1,y_1)}$ through the resolvent R. Let $\tilde{\chi}_{(x_1,y_1)}$ be a smooth characteristic function of the unit cube centered at (x_1, y_1), so that $\tilde{\chi}_{(x_1,y_1)}\mathbf{1}_{(x_1,y_1)} = \mathbf{1}_{(x_1,y_1)}$. We get

$$\|R_{>R}\mathbf{1}_{(x_1,y_1)}WRR_{>R}[H,\mathcal{X}]\mathbf{1}_{(x_2,0)}\|_1 \tag{4.19}$$
$$\leq \|R_{>R}\mathbf{1}_{(x_1,y_1)}WR\tilde{\chi}_{(x_1,y_1)}R_{>R}[H,\mathcal{X}]\mathbf{1}_{(x_2,0)}\|_1 \tag{4.20}$$
$$+\|R_{>R}\mathbf{1}_{(x_1,y_1)}WR[H,\tilde{\chi}_{(x_1,y_1)}]RR_{>R}[H,\mathcal{X}]\mathbf{1}_{(x_2,0)}\|_1\,. \tag{4.21}$$

The term in (4.20) is estimated as previously, as for the one in (4.21) note that it follows from Lemma 3 Eq. A.8 and the Combes-Thomas estimate (A.4) that

$$\|[H,\tilde{\chi}_{(x_1,y_1)}]RR_{>R}[H,\mathcal{X}]\mathbf{1}_{(x_2,0)}\| \tag{4.22}$$
$$\leq \sum_{(x_3,y_3)\in\mathbb{R}^3} \|[H,\tilde{\chi}_{(x_1,y_1)}]R\mathbf{1}_{(x_3,y_3)}\|\|\mathbf{1}_{(x_3,y_3)}R_{>R}[H,\mathcal{X}]\mathbf{1}_{(x_2,0)}\| \tag{4.23}$$
$$\leq \frac{C}{\eta^3}(\langle x_1\rangle + \langle y_1\rangle)\langle x_2\rangle \mathrm{e}^{-c\eta(|x_2-x_1|+|y_1|)}\,. \tag{4.24}$$

The rest of the argument follows as above. It allows us to conclude that

$$\left\|(R_{>R}(u+iv)^2 - R(u+iv)^2)i[H,\mathcal{X}]\right\|_1 \leq \frac{C(L_2-L_1)\|W\|_\infty}{\eta^\kappa}\mathrm{e}^{-c\eta R}, \tag{4.25}$$

for some integer κ. Following (A.2), it remains to integrate the latter estimate multiplied by $|\bar{\partial}\tilde{g}_n(z)|$, $z = u + iv$. By Lemma 2, it follows that for any integer $m \geq 1$ there exists C_m such that for any $R \geq 1$,

$$\|(g'(H + W_{>R}) - g'(H))i[H,\mathcal{X}]\|_1 \leq C_m R^{-m}. \tag{4.26}$$

So (2.6) holds, and (2.7) is a direct consequence of (2.6) and Proposition 1. □

5. Estimating differences of a priori non finite edge conductances

This section is devoted to the proof of Theorem 2

Proof. The main task is to prove (2.10), and that the operator coming in is trace class. Assuming this, let us sketch how to derive the second part of the statement, and in particular (2.11). If V_0 is a confining potential, then, with the abuse of notations of Footnote 4, it follows from (2.10) that (in addition to the trace class property)

$$\mathrm{tr}((g'(H(V_1,V_0)) - g'(H(V_1,0)) - g'(H(0,V_0)))i[H_L,\mathcal{X}]) = 0\,, \tag{5.1}$$
$$\mathrm{tr}((g'(H(V_0,V_2)) - g'(H(V_0,0)) - g'(H(0,V_2)))i[H_L,\mathcal{X}]) = 0\,, \tag{5.2}$$
$$\mathrm{tr}((g'(H(V_0,0)) + g'(H(0,V_0)) - g'(H(V_0,V_0)))i[H_L,\mathcal{X}]) = 0\,. \tag{5.3}$$

Substract these equations to (2.10) and note that, V_0 being confining, Corollary 2 implies that $g'(H(V_0,V_0)))i[H_L,\mathcal{X}]$ is trace class with trace zero. This yields the announced (2.11).

We now prove the first part of the statement. For $R \geq 0$, set

$$\begin{aligned} D(R) = \{ \;& g'(H(V_1,V_2)) - g'(H(0,V_2)) \\ & -g'(H(V_1,V_2\mathbf{1}_{x\leq R})) + g'(H(0,V_2\mathbf{1}_{x\leq R}))\}i[H_L,\mathcal{X}] \end{aligned} \tag{5.4}$$

Since $g'(H(0,0)) = 0$ (suppg' is included in a gap of $H(0,0) = H_L$), (2.10) of the theorem is proved if we show that $D(0)$ is trace class with trace zero. Now, that $D(R) - D(0)$ is trace class with trace zero is an immediate consequence of Theorem 1. It is thus enough to show that $D(R)$ is trace class and that $\lim_{R\to+\infty} |\mathrm{tr} D(R)| = 0$.

As previously we use the Helffer-Sjöstrand functional calculus to write operators of the type $g'(H)$ in term of second power of resolvents, and then make use of the second order resolvent equation (4.3). We shall make use of the following notations: $H = H(V_1,V_2)$, $H_2 = H(0,V_2)$, as for the operators with a truncated V_2 we set $H_{\leq R} = H(V_1, V_2\mathbf{1}_{x\leq R})$, $H_{2,\leq R} = H(0, V_2\mathbf{1}_{x\leq R})$; with respective resolvents R, R_2, $R_{\leq R}$, $R_{2,\leq R}$. We get

$$\begin{aligned} &(R^2 - R_2^2) - (R_{\leq R}^2 - R_{2,\leq R}^2) = \\ &\quad -RR_2V_1R - R_2V_1RR_2 + R_{\leq R}R_{2,\leq R}V_1R_{\leq R} + R_{2,\leq R}V_1R_{\leq R}R_{2,\leq R}. \end{aligned} \tag{5.5}$$

We first treat the term $RR_2V_1R - R_{\leq R}R_{2,\leq R}V_1R_{\leq R}$. Bounding the remaining one will be done in a similar way, and it is discussed below. since $H - H_{\leq R} = H_2 - H_{2,\leq R} = V_2\mathbf{1}_{x>R} \equiv V_{2,>R}$, one has

$$RR_2V_1R - R_{\leq R}R_{2,\leq R}V_1R_{\leq R} = -RR_2V_1RV_{2,>R}R_{\leq R} \tag{5.6}$$
$$-RR_2V_{2,>R}R_{2,\leq R}V_1R_{\leq R} - RV_{2,>R}R_{\leq R}R_{2,\leq R}V_1R_{\leq R}. \tag{5.7}$$

Let us first prove that $\|RR_2V_1RV_{2,>R}R_{\leq R}[H,\mathcal{X}]\|_1$ decays faster than any polynomial in R. With $X_i = (x_i,y_i)$, $i = 1,2$, write $V_1 = \sum_{X_1\in S_1} V_1\mathbf{1}_{X_1}$ with $S_1 = \mathbb{Z}^- \times \mathbb{Z}$, $V_{2,>R} = \sum_{X_2\in S_2} V_2\mathbf{1}_{X_2}$ with $S_2 = (\mathbb{Z}\cap]R,+\infty[) \times \mathbb{Z}$, and

$[H,\mathcal{X}] = \sum_{x_3\in\mathbb{Z}}[H,\mathcal{X}]\mathbf{1}_{(x_3,0)}$ as in Section 4, Proof of Theorem 1. Then, with $\mathbf{1}_i = \mathbf{1}_{X_i}$, $i=1,2$, and κ some integer that will vary from one line to another:

$$\begin{aligned}
&\|RR_2V_1RV_{2,>R}R_{\le R}[H,\mathcal{X}]\|_1\\
&\quad\le \sum_{\substack{(x_1,y_1)\in S_1\\(x_2,y_2)\in S_2\\x_3\in\mathbb{Z}}} \|RR_2V_1\mathbf{1}_1\|_1\|\mathbf{1}_1R\mathbf{1}_2\|\|\mathbf{1}_2V\|\|\mathbf{1}_2R_{\le R}[H,\mathcal{X}]\mathbf{1}_{(x_3,0)}\|\\
&\quad\le \sum_{\substack{(x_1,y_1)\in S_1\\(x_2,y_2)\in S_2\\x_3\in\mathbb{Z}}} \frac{C}{\eta^\kappa}\|\mathbf{1}_1V_1\|\,\|\mathbf{1}_2V\|\mathrm{e}^{-\eta(|x_1-x_2|+|y_1-y_2|+|x_2-x_3|+|y_2|)}\\
&\quad\le \frac{C\|V\|_\infty}{\eta^\kappa}\sum_{\substack{x_1\in\mathbb{Z}^-\\x_2\in\mathbb{Z}\cap]R,+\infty[}} \|\mathbf{1}_{(x_1,0)}V_1\|\mathrm{e}^{-\eta|x_1-x_2|}\\
&\quad\le \frac{C\|V\|_\infty}{\eta^\kappa}\sum_{x_1\in\mathbb{Z}^-} \|\mathbf{1}_{(x_1,0)}V_1\|\mathrm{e}^{-\eta(|x_1|+R)}. \qquad (5.8)
\end{aligned}$$

We used Lemma 4, the Combes-Thomas estimate (A.4), as well as Lemma 3. We also used the invariance of V_1 in the y-direction. Since by Assumption 2.2 we have the bound $\|\mathbf{1}_{(x_1,0)}V_1\| \le C\langle x_1\rangle^p$, for some $p<\infty$, it follows from (5.8) that for some constant C and integer $\kappa>0$ (depending on p) that

$$\|RR_2V_1RV_{2,>R}R_{\le R}[H,\mathcal{X}]\|_1 \le \frac{C\|V\|_\infty}{\eta^\kappa}\mathrm{e}^{-\eta R} \qquad (5.9)$$

The second term coming from (5.7) is estimated exactly as the first one. The third contribution from (5.7) requires an extra argument. If one is only interested in the decay (in R) of its trace, and not of its trace norm, then the above argument applies again if one notices that by cyclicity $\mathrm{tr}(RV_{2,>R}R_{\le R}R_{2,\le R}V_1R_{\le R}[H,\mathcal{X}]) = \mathrm{tr}(R_{\le R}R_{2,\le R}V_1R_{\le R}[H,\mathcal{X}]RV_{2,>R})$.

Let us now briefly comment how to control the remaining contribution from (5.5), that is the one coming from the difference $R_2V_1RR_2 - R_{2,\le R}V_1R_{\le R}R_{2,\le R}$. One first decomposes it in three terms as in (5.7). To get the decay of the trace of each of the three contribution one can use cyclity of the trace and apply the argument above. These estimates lead to $|\mathrm{tr}D(R)| \le C_mR^{-m}$ for any $m>0$. □

We note that actually the stronger $\|D(R)\|_1 \le C_mR^{-m}$ for any $m>0$ can be proven. It is indeed sufficient to use a similar argument to the one given in (4.19) and subsequent.

6. The Nakamura-Bellissard model revisited

In [NB] Nakamura and Bellissard showed that the bulk Hall conductance σ_b vanishes in any Landau band for sufficiently large coupling constant in a positive potential exhibiting non degenerate wells locally identical (e.g. a periodic potential). Their proof is based on semi-classical analysis at large coupling and non commutaive geometry methods. It turns out that the vanishing of σ_e can be obtained in a simple way from Theorem 1.

Assume that the bulk potential V_b satisfies the assumptions of [NB]. Namely and with irrelevant simplifications (we set $X = (x, y)$):
(i) $\inf V_b(X) = 0$ and $\sup V_b(X) < \infty$;
(ii) there is a countable set $\{X_n,\ n = 1, 2, \cdots\}$ such that one has $|X_n - X_m| \geq 1$ if $n \neq m$;
(iii) V_b has identical potential wells located at the X_n's, i.e., there exists $\varepsilon \in]0, \frac{1}{2}[$, and $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^2)$, such that for all $n = 1, 2, \cdots$, $V_b(X + X_n) = \mathcal{V}(X)$ if $|X| \leq \varepsilon$;
(iv) 0 is the unique minimum of $\mathcal{V}$ and it is non degenerate;
(v) if $|X - X_n| > \varepsilon$ for all n, then $V_b(X) > \delta$, for some $\delta > 0$.

Then by a semi-classical analysis patterned according to the method developped in [BCD], it is shown that for large μ then spectrum of $H_b(\mu) = H_L + \mu V_b$ consists, in the range $] - \infty, \mu^{\frac{1}{2}}[$, of bands $\mathcal{B}_{n,m}$ centered around the eigenvalues $E_{n,m}(\mu)$ of the one well Hamiltonian

$$h(\mu) = H_L + \mu \mathcal{V}, \tag{6.1}$$

which, in the large μ regime, satisfies the harmonic approximation:

$$E_{n,m}(\mu) = \mu^{\frac{1}{2}}((n+1)W_1 + (m+1)W_2) + \mathcal{O}(1), \tag{6.2}$$

where $W_{1,2}$ are the eigenvalues of the Hessian of $\mathcal{V}$ at $x = 0$. The bands $\mathcal{B}_{n,m}$ have width

$$\Delta_{n,m}(\mu) < \mathrm{e}^{-a\mu^{\frac{1}{2}}}, \tag{6.3}$$

where a is a lower bound on Agmon's distance between different wells (see Theorem 6.1 in [NB]). So everything only depends on ε and δ. This implies that this spectral structure is not changed under the following modifications of V_b:
a) fill the well at X_n up to δ if $X_n \in S_1 = \{(x, y), |x| < 1\}$;
b) replace V_b in the half plane $\{x < 0\}$ by some constant potential $v_0 > \delta$.

Accordingly if $I \subset]B_N, B_{N+1}[$, $N \geq 0$, satisfies $\mathrm{dist}(I, \sigma(h(\mu))) > \mathrm{e}^{-a\mu^{\frac{1}{2}}}$ and $\sup I < \mu\delta$, then for μ large enough, I is in a gap of

$$H_e(\mu) := H_L + \mu(v_0 \mathbf{1}_- + V_b \mathbf{1}_+ + W), \tag{6.4}$$

where

$$W(X) = \sum_{X_n \in S_1 \setminus S_0} (\delta - \mathcal{V}(X - X_n)) \mathbf{1}_{|X - X_n| \leq \varepsilon}(X). \tag{6.5}$$

So, as long as $\mathrm{supp} g' \subset I$, one has $\sigma_e(g, H_e(\mu)) = 0$, and according to Theorem 1 one also obtains

$$\sigma_e(g, \mu v_0, \mu V_b) = 0.$$

This is the "edge picture" of [NB]'s result. Indeed equality of bulk and edge conductances then yields that the bulk conductance is zero if the Fermi energy belongs to I, which is [NB]'s result. Moreover in virtue of Theorem 2 this in turn implies that

$$\sigma_e(g, 0, \mu V_b) = -N,$$

and thus that $H_L + \mu V_b \mathbf{1}_+$ has current carrying edge states for large μ.

7. Regularizing the edge conductance in presence of impurities

Let V be a potential located in the region $x \geq 0$. If the operator $H(0,V)$ has a gap and if the interval I falls into this gap, then the edge conductance is quantized by Corollary (4). A more challenging issue is to show quantization if I falls into a region of localized states of $H(0,V)$. In the latter case, conductances may not be well-defined, and a regularization is needed. This is the content of this section. We propose some basic conditions that a "good" regularization should fulfill and discuss some candidates.[5].

Let V_0 be a y-invariant left confining potential with respect to $I = [a,b] \subset]B_N, B_{N+1}[$, and assume supp$g' \subset I$. Let $(J_R)_{R>0}$ be a family of operators s.t.

C1. $\|J_R\| = 1$ and $\lim_{R\to\infty} J_R\psi = \psi$ for all $\psi \in E_{H(0,V)}(I)\mathrm{L}^2(\mathbb{R}^2)$.
C2. J_R regularizes $H(0,V)$ in the sense that $g'(H(0,V))i[H_L,\mathcal{X}]J_R$ is trace class for all $R>0$, and $\lim_{R\to\infty}\mathrm{tr}(g'(H(0,V))i[H_L,\mathcal{X}]J_R)$ exists and is finite.

Then it follows from Corollary 3 that

$$\lim_{R\to\infty} -\mathrm{tr}\left(\{g'(H(V_0,V)) - g'(H(0,V))\}i[H_L,\mathcal{X}]J_R\right) = N.$$

In other terms, if **C1** and **C2** hold, then J_R also regularizes $H(V_0,V)$. Defining the regularized edge conductance by

$$\sigma_e^{\mathrm{reg}}(g,V_1,V_2) := -\lim_{R\to\infty}\mathrm{tr}(g'(H(V_1,V_2))i[H_L,\mathcal{X}]J_R), \tag{7.1}$$

whenever the limit exists, we get the analog of Corollary 3:

$$\sigma_e^{\mathrm{reg}}(g,V_0,V) = N + \sigma_e^{\mathrm{reg}}(g,0,V). \tag{7.2}$$

In particular, if we can show that $\sigma_e^{\mathrm{reg}}(g,0,V) = 0$, for instance under some localization property, then the edge quantization for $H(V_0,V)$ follows:

$$\sigma_e^{\mathrm{reg}}(g,V_0,V) = -\lim_{R\to\infty}\mathrm{tr}(g'(H(V_0,V))i[H_L,\mathcal{X}]J_R) = N. \tag{7.3}$$

To start the discussion, consider as the simplest candidate for J_R, the multiplication by the characteristic function of the half plane $x < R$ (or a smooth version of it). One checks that **C1** holds and that the trace class condition in **C2** is fulfilled (to see this consider the difference $\{g'(H(0,V)) - g'(H(0,0))\}i[H,\mathcal{X}]J_R$ and proceed as in the proof of Theorem 1). As for the limit $R\to\infty$ of the trace in **C2**, we do not expect it to exist in full generality. However, if $H_\omega = H(0,V_{\omega,+})$ is a random operator with i.i.d. variables, then it follows from our previous results that the limit exists. Indeed, consider

$$H_\omega = H(0,V_{\omega,+}) = H_L + V_{\omega,+}, \quad V_{\omega,+} = \sum_{i\in\mathbb{Z}^{+*}\times\mathbb{Z}} \omega_i u(x-i), \tag{7.4}$$

a random operator modeling impurities located on the positive half plane (the $(\omega_i)_i$ are i.i.d. random variables, and u is a bump function). The following proposition shows that the current flowing far from the edge $x=0$ is negligible (in the expectation sense).

[5] In [EGS], related questions are adressed. We thus also refer the reader to their preprint

Proposition 2. *Let $H_\omega = H(0, V_{\omega,+})$ as in (7.4), and $J_R = \mathbf{1}_{x\le R}$. For all $p \in \mathbb{N}^*$, there exists $C_p > 0$ finite, such that, for all $R > 0$,*

$$|\mathbb{E}\left(\mathrm{tr}\left\{g'(H_\omega) i[H_L, \mathcal{X}](J_{R+1} - J_R)\right\}\right)| \le C_p R^{-p}\,. \tag{7.5}$$

*As a consequence, for $\mathbb{P}$-a.e. ω, $\lim_{R\to\infty} \mathrm{tr}(g'(H_\omega)i[H_L,\mathcal{X}]J_R)$ exists and is finite. In other terms, for $\mathbb{P}$-a.e. ω, J_R satisfies **C1** and **C2** and the rule (7.2) holds. Moreover, if H_ω has pure point spectrum in I for $\mathbb{P}$-a.e. ω, then denoting by $(\varphi_{\omega,n})_{n\ge1}$ a basis of orthonormalized eigenfunctions of H_ω with energies $E_{\omega,n} \in \mathrm{supp} g' \subset I$, one has*

$$\sigma_e^{\mathrm{reg}}(g, 0, V_{\omega,+}) = -\lim_{R\to\infty} \sum_n g'(E_{\omega,n}) \langle \varphi_{\omega,n}, i[H_\omega, \mathcal{X}] J_R \varphi_{\omega,n}\rangle. \tag{7.6}$$

Proof. Let H_ω^1 be obtained from H_ω by setting $\omega_i = 0$ for all $i \in \{1\} \times \mathbb{R}$. The random variables ω_i being i.i.d., one has

$$\mathbb{E}\left(\mathrm{tr}\left\{g'(H_\omega)i[H_L,\mathcal{X}]J_R\right\}\right) = \mathbb{E}\left(\mathrm{tr}\left\{g'(H_\omega^1)i[H_L,\mathcal{X}]J_{R+1}\right\}\right)\,. \tag{7.7}$$

Moreover since the operator $H_\omega^1 - H_\omega$ leaves in a vertical strip of finite width, it follows by Theorem 1 that

$$\mathbb{E}\left(\mathrm{tr}\left\{(g'(H_\omega) - g'(H_\omega^1))i[H_L,\mathcal{X}]\right\}\right) = 0\,. \tag{7.8}$$

On the other hand, using arguments as in the proof of Theorem 1, one has that for any $p > 0$ there exists $C_p < \infty$ s.t.

$$\left|\mathbb{E}\left(\mathrm{tr}\left\{(g'(H_\omega) - g'(H_\omega^1))i[H_L,\mathcal{X}](1 - J_{R+1})\right\}\right)\right| \le C_p R^{-p}\,. \tag{7.9}$$

By (7.8) and (7.9),

$$\left|\mathbb{E}\left(\mathrm{tr}\left\{g'(H_\omega)i[H_L,\mathcal{X}]J_{R+1}\right\}\right) - \mathbb{E}\left(\mathrm{tr}\left\{g'(H_\omega^1)i[H_L,\mathcal{X}]J_{R+1}\right\}\right)\right| \le C_p R^{-p}\,. \tag{7.10}$$

Plugging (7.7) into the latter yields (7.5). The expression in (7.6) follows by expanding the trace other the basis of eigenfunctions. □

However although the limit exists it is very likely that the quantity in (7.6) will not be zero, even under strong localization properties of the eigenfunctions such as (SULE) (see [DRJLS]) or (WULE) (see Definition 2 below)[6]. This can be understood from the fact that the frontier of $J_R = \mathbf{1}_{x\le R}$ intersects classical orbits, creating thereby spurious contributions to the total current. The quantum counter part of this picture is that although the expectation of $i[H(0,V),\mathcal{X}]$ in an eigenstate of $H(0,V)$ is zero by the Virial Theorem this is not true anymore if this commutator is multiplied by $J_R = \mathbf{1}_{<R}$. Of course if J_R commutes with $H(0,V)$ then the sum in (7.6) is zero.

One way to prevent spurious contributions to the current, is to select the eigenfunctions living in the region $\{x \le R\}$, rather than multiplying the velocity term $i[H_L,\mathcal{X}]$ by $\mathbf{1}_{\le R}$. In (7.13) below we shall introduce the regularization $J_R = \sum_{E_n\in I} E_{H(0,V)}(\{E_n\})\mathbf{1}_{x\le R}E_{H(0,V)}(\{E_n\})$. Roughly, it yields a factor of the form $\langle\varphi_{n,m}, \mathbf{1}_{x\le R}\varphi_{n,m}\rangle$ that is small for eigenfunctions $\varphi_{n,m}$ living far from the region $\{x \le R\}$. What we need is therefore (i) a sufficient decay in $|X - X'|$ of

[6] We note that a similar quantity appears in [EGS].

$\|\mathbf{1}_X E_{H(0,V)}(\{E_n\})\mathbf{1}_{X'}\|$ and (ii) a summability condition over E_n's in I. Such a signature of localization has been discussed in [Ge], and has been called (WULE), for Weakly Uniformly Localized Eigenfunctions.

Let $T(X)=(1+|X|^2)^\nu$, $\nu>d/4$. It is well known for Schrödinger operators that $\mathrm{tr}(T^{-1}E_{H(0,V)}(I)T^{-1})<\infty$, if I is compact (e.g. [KKS, GK3]). We set

$$\mu(J):=\mathrm{tr}(T^{-1}E_{H(0,V)}(J\cap I)T^{-1})<\infty. \tag{7.11}$$

Definition 2 (WULE). *Assume $H(0,V)$ has pure point spectrum in I with eigenvalues E_n. Let μ be the measure defined in* (7.11)*. We say that $H(0,V)$ has (WULE) in I, if there exist a mass $\gamma>0$ and a constant C such that for any $E_n\in I$ and $X_1,X_2\in\mathbb{Z}^2$,*

$$\|\mathbf{1}_{X_1}E_{H(0,V)}(\{E_n\})\mathbf{1}_{X_2}\|\le C\mu(\{E_n\})\|T\mathbf{1}_{X_1}\|\|T\mathbf{1}_{X_2}\|\mathrm{e}^{-\gamma|X_1-X_2|}. \tag{7.12}$$

Remark 10. The measure μ in (7.11) is the one that appears in the Generalized Eigenfunctions Expansion (GEE) as in [Si, KKS], its kernel being given by $P_\lambda:=E_{H(0,V)}(\{\lambda\})/\mu(\{\lambda\})$. So (7.12) asserts that $\|\mathbf{1}_{X_1}P_{E_n}\mathbf{1}_{X_2}\|$ decays exponentially. We further note that alternatively to (7.12), one could assume that $\|\mathbf{1}_{X_1}\varphi_{n,m}\|_{\mathrm{L}^2}\|\mathbf{1}_{X_2}\varphi_{n,m}\|_{\mathrm{L}^2}\le C\|T^{-1}\varphi_{n,m}\|^2\|T\mathbf{1}_{X_1}\|\|T\mathbf{1}_{X_2}\|\mathrm{e}^{-\gamma|X_1-X_2|}$, with the $(\varphi_{n,m})$'s being an orthonormalized basis of eigenfunctions of eigenvalue $E_n\in I$.

Theorem 3. *Assume that $H(0,V)$ has (WULE) in I. Then*

$$J_R=\sum_{E_n\in I}E_{H(0,V)}(\{E_n\})\mathbf{1}_{x\le R}E_{H(0,V)}(\{E_n\}) \tag{7.13}$$

*regularizes $H(0,V)$, and thus also $H(V_0,V)$, in the sense that **C1** and **C2** hold. Moreover the edge conductances are quantized, and one has: $\sigma_e^{\mathrm{reg}}(g,0,V)=0$ and $\sigma_e^{\mathrm{reg}}(g,V_0,V)=N$ if $I\subset]B_N,B_{N+1}[$ for some $N\ge 0$.*

Remark 11. An other possible regularization is to use the stronger localization signature called (SULE) introduced in [DRJLS] (see also [GDB, GK1]). It requires an exponential decay of the eigenfunctions of the form $\|\mathbf{1}_X\varphi_n\|_{L^2}\le Ce^{(\log|X_n|)^2}e^{-\gamma|X-X_n|}$ with centers of localization $X_n=(x_n,y_n)\in\mathbb{Z}^2$. Then one can show that $J_R=\sum_{x_n\le R}|\varphi_n\rangle\langle\varphi_n|$ satisfies **C1** and **C2**, with in addition $\sigma_e^{\mathrm{reg}}(g,0,V)=0$ and $\sigma_e^{\mathrm{reg}}(g,V_0,V)=N$ if $I\subset]B_N,B_{N+1}[$, $N\ge 0$.

Remark 12. Let $H(0,V_{\omega,+})=H_L+V_{\omega,+}$ be a random operator as in (7.4) and hypotheses on u and the ω_i's are as in [CH, Wa, GK3] (also [DMP]). It can be noted that the percolation estimates due to [CH, Wa] are still effective in the region where the potential is zero. The Wegner estimate given in [CH] is insensitive to this modification as well. Since for energies away from the Landau levels no eigenfunction can live in the left region, it is natural to expect a modified version of the multiscale analysis performed in [CH, Wa, GK3] to hold (or equivalently a version of the fractional moment method developed in [AENSS] if the support of the single bump u covers a unit cube). This is done in [CGH] where localization is proved away from the Landau levels. In particular the following result holds true: For $N\in\mathbb{N}$, there exists constants K_N (depending on the parameters of the model, except B), so that for B large enough, and if g is s.t. $\mathrm{dist}(\mathrm{supp}g',\{B_N,B_{N+1}\})\ge K_N\frac{\log B}{B}$ for some $N\ge 0$, then $H(0,V_{\omega,+})$ has (WULE) in I for $\mathbb{P}$-a.e. ω and Theorem 3 applies.

Proof. To show **C1**, note that for all $\phi \in \mathcal{H}$ and $A \subset \mathbb{R}^2$:

$$\left\| \sum_{E_n \in I} E_{H(0,V)}(\{E_n\}) \mathbf{1}_A E_{H(0,V)}(\{E_n\}) \phi \right\|^2 \tag{7.14}$$

$$\leq \sum_{E_n \in I, m \geq 1} \|\mathbf{1}_A \varphi_{n,m}\|^2 \, |\langle \varphi_{n,m}, \phi \rangle|^2 \leq \|\phi\|^2. \tag{7.15}$$

where $(\varphi_{n,m})_{m\geq 1}$ denotes an orthonormalized basis of eigenfunctions of energy $E_n \in I$. With $A = \{x \leq R\}$ the last bound yields $\|J_R\| \leq 1$. Next, use the first bound in (7.15) with $A = \{x > R\}$ together with the Lebesgue Dominated Convergence Theorem to get that $J_R \to E_{H(0,V)}(I)$.

We turn to **C2**. Write $[H_L, \mathcal{X}] = \sum_{x_2 \in \mathbb{Z}} [H_L, \mathcal{X}] \mathbf{1}_{(x_2,0)}$ as in Section 4. We get

$$\begin{aligned} &\|g'(H(0,V)) i [H_L, \mathcal{X}] J_R\|_1 \\ &\leq \sum_{E_n \in I} \sum_{X_1, x_2} \left\| g'(H(0,V)) i [H_L, \mathcal{X}] \mathbf{1}_{(x_2,0)} E_{H(0,V)}(\{E_n\}) \mathbf{1}_{X_1} E_{H(0,V)}(\{E_n\}) \right\|_1 \\ &\leq C \sum_{E_n \in I} \sum_{X_1, x_2} \|\mathbf{1}_{(x_2,0)} E_{H(0,V)}(\{E_n\}) \mathbf{1}_{X_1}\|_2 \|\mathbf{1}_{X_1} E_{H(0,V)}(\{E_n\})\|_2, \end{aligned} \tag{7.16}$$

where the summation is over X_1's s.t. $x_1 \leq R$; in the last bound we used that $\|g'(H(0,V)) i [H_L, \mathcal{X}]\| \leq C$. Next, the exponential decay due to (7.12) carries over to Hilbert-Schmidt operator kernels, since

$$\|\mathbf{1}_{X_1} E_{H(0,V)}(\{E_n\}) \mathbf{1}_{X_2}\|_2^2 \leq \mu(I) \|T \mathbf{1}_{X_1}\| \|T \mathbf{1}_{X_2}\| \|\mathbf{1}_{X_1} E_{H(0,V)}(\{E_n\}) \mathbf{1}_{X_2}\|.$$

It ensures that for any given $x_1 \leq R$, the sum over $y_1, x_2 \in \mathbb{Z}$ converges, while $\sum_n \mu(\{E_n\}) = \mu(I) < \infty$ takes care of the summation over n. To complete the argument it is thus enough to show summability in $x_1 \leq -2$. This will come from the fact that eigenfunctions cannot live far inside the region $\{x \leq 0\}$. More precisely, for $x_1 \leq -2$, let Λ be a box centered at $X_1 = (x_1, y_1)$ and of radius $|x_1| - 1$, and $\tilde{\mathbf{1}}_\Lambda$ be a smooth version of $\mathbf{1}_\Lambda$, s.t. $\tilde{\mathbf{1}}_\Lambda V \mathbf{1}_+ = 0$. Pass $\tilde{\mathbf{1}}_\Lambda$ through $H(0,V) = H_L + V\mathbf{1}_+$ in $\tilde{\mathbf{1}}_\Lambda (H(0,V) - E_n) E_{H(0,V)}(\{E_n\}) = 0$; multiply on the left by $\mathbf{1}_{X_1}(H_L - E_n)^{-1}$; use Combes-Thomas to control the resolvent of H_L. It follows that $\|\mathbf{1}_{X_1} E_{H(0,V)}(\{E_n\})\|$ decays exponentially in $|x_1|$[7]. □

Remark 13. Notice that the J_R of (7.13) considered in Theorem 3 also reads

$$J_R = s - \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{e}^{itH(0,V)} E_{H(0,V)}(I) \mathbf{1}_{x \leq R} E_{H(0,V)}(I) \mathrm{e}^{-itH(0,V)} \mathrm{d}t. \tag{7.17}$$

We expect that one can contruct a regularization in the spirit of (7.17), assuming only that $H(0,V)$ exhibits *dynamical localization* [A,GDB,GK1] in I.[8]

[7] An alternetive to this last step is to exploit the decay in the region $\{x \leq 0\}$ coming from $g'(H(0,V)) = g'(H(0,V)) - g'(H(0,0))$.

[8] Note that the form (7.17) is close to the regularization considered in [EGS].

A. Appendix A: Some decay estimates

For $g \in \mathcal{C}_c^\infty(\mathbb{R})$, let $\tilde{g}_n$ be a quasi-analytic extension of g of order $n \geq 1$ of the form

$$\tilde{g}_n(u+iv) = \rho(u,v) S_n\tilde{g}(u+iv), \qquad S_n\tilde{g}(u+iv) = \sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(u)(iv)^k, \quad \text{(A.1)}$$

where $\rho(u,v) = \tau(v/\langle u\rangle)$; the function τ is smooth such that $\tau(t) = 1$ for $|t| \leq 1$ and $\tau(t) = 0$ for $|t| \geq 2$. For H as in (2.1), the Helffer-Sjöstrand formula [HeSj, HuSi] reads

$$g'(H) = -\frac{1}{\pi} \int \bar{\partial}\tilde{g}_n(u+iv)(H-u-iv)^{-2} \mathrm{d}u\mathrm{d}v, \qquad \bar{\partial} = \frac{1}{2}\left(\partial_u + i\partial_v\right). \quad \text{(A.2)}$$

One has $\bar{\partial}\tilde{g}_n(u+iv) = (\bar{\partial}\rho(u,v))S_n\tilde{g}(u+iv) + \rho(u,v)\bar{\partial}S_n\tilde{g}(u+iv)$. But a simple computation yields: $\bar{\partial}(S_n\tilde{g})(u+iv) = \frac{1}{2n!} g^{(n+1)}(u)(iv)^n$. As a consequence,

$$\bar{\partial}\tilde{g}_n(u+iv) = \bar{\partial}\rho(u,v) \sum_{k=0}^{n} \frac{1}{k!} g^{(k)}(u)(iv)^k + \frac{\rho(u,v)}{2}\frac{1}{n!} g^{(n+1)}(u)(iv)^n. \quad \text{(A.3)}$$

Since u takes values in $\mathrm{supp} g'$ compact, the usual Combes-Thomas estimate is sufficient for our purpose [CT], namely,

$$\|\mathbf{1}_x (H-z)^{-1}\mathbf{1}_y\| \leq \left(\tfrac{C}{\eta}\right) \exp\left(-c\eta|x-y|\right), \quad \eta = \mathrm{dist}(u+iv, \sigma(H)), \quad \text{(A.4)}$$

with constants $C, c > 0$ depending on g. In practice, (A.4) will be used in combination with Lemma 4 and Lemma 3. To conclude we shall use the following lemma.

Lemma 2. *Let H and g be as above, $\tilde{g}$ be the quasi-analytic extension of g to the order n given by* (A.1)*, and $\eta = \mathrm{dist}(u+iv, \sigma(H))$. Let $f_{L,\kappa}(\eta) = \eta^{-\kappa}\mathrm{e}^{-c\eta L}$ for some $\kappa \geq 0$ and $L > 0$. For any $m \geq 1$, if $n \geq m+\kappa$, there exists a constant c depending only on n, m, κ and on g (through its support and $\|g^k\|_\infty$, $k = 0, 1, \cdots, n+1$), such that*

$$\int \left|\bar{\partial}\tilde{g}_n(u+iv)\right| f_{L,\kappa}(\eta) \mathrm{d}u\mathrm{d}v \leq \frac{c}{L^m}. \quad \text{(A.5)}$$

Remark 14. If g is chosen to be Gevrey of class $a > 1$, then following [BGK] the integral in (A.5) decays sub-exponentially like $\exp(-cL^{1/a'})$ with any $a' > a$.

Lemma 3. *Let χ_1 and χ_2 be two smooth functions localized on compact regions of $\mathbb{R}^2$. Let $\tilde{\chi}_2$ be a smooth function s.t. $\tilde{\chi}_2 = 1$ on the support of χ_2, and denote by $R(z)$ the resolvent of $H_L + V = \Pi_x^2 + \Pi_y^2 + V$. Then, with α standing for either x or y,*

$$\begin{aligned} &\|\chi_1 R(z)\Pi_\alpha \chi_2\|^2 \qquad \text{(A.6)} \\ &\leq 2(|z| + \|V\|_\infty + 2\|p_x\chi_2\|_\infty^2 + 4\|p_y\chi_2\|_\infty^2 + \|Bx\chi_2\|_\infty^2)\|\chi_1 R(z)\tilde{\chi}_2\|^2 \\ &\quad +2\|\chi_1\tilde{\chi}_2\|_\infty \|\chi_1 R(z)\tilde{\chi}_2\|. \end{aligned}$$

As a consequence, let $\tilde{\mathcal{X}}$ be a smooth function equal to 1 on the support of $\mathcal{X}'$ (typically, $\tilde{\mathcal{X}} = 1$ on $\mathbb{R} \times [-\frac{1}{4}, \frac{1}{4}]$, and $\tilde{\mathcal{X}} = 0$ outside $\mathbb{R} \times [-\frac{1}{2}, \frac{1}{2}]$), then

$$\begin{aligned} &\|\chi_1 R(z)[H, \mathcal{X}]\chi_2\|^2 \\ &\quad \leq (C + 2|z| + 2B\|x\chi_2\mathcal{X}\|_\infty^2)\|\chi_1 R(z)\tilde{\mathcal{X}}\tilde{\chi}_2\|^2 + 2\|\chi_1\tilde{\mathcal{X}}\chi_2\|_\infty \|\chi_1 R(z)\tilde{\mathcal{X}}\chi_2\|, \end{aligned} \tag{A.7}$$

where C depends on V, $\mathcal{X}'$, $\mathcal{X}''$, $\tilde{\mathcal{X}}$ and χ_2 as in (A.6)*, i.e. through their sup norm. In particular, if the supports of χ_1 and $\tilde{\mathcal{X}}\chi_2$ are disjoints, one has*

$$\|\chi_1 R(z)[H, \mathcal{X}]\chi_2\| \leq (C + 2|z| + 2B\|x\chi_2\mathcal{X}\|_\infty^2)^{\frac{1}{2}} \|\chi_1 R(z)\tilde{\mathcal{X}}\tilde{\chi}_2\| \,. \tag{A.8}$$

Proof. We have to bound $\|\chi_2 \Pi_\alpha R(\overline{z})\chi_1\varphi\|^2$, with $\varphi \in \mathcal{C}_c^\infty$. We get

$$\begin{aligned} &\|\chi_2 \Pi_\alpha R(\overline{z})\chi_1\varphi\|^2 \\ &= \langle R(\overline{z})\chi_1\varphi, \Pi_\alpha \chi_2^2 \Pi_\alpha R(\overline{z})\chi_1\varphi\rangle \\ &= \langle R(\overline{z})\chi_1\varphi, (\Pi_\alpha \chi_2^2)\Pi_\alpha R(\overline{z})\chi_1\varphi\rangle + \langle R(\overline{z})\chi_1\varphi, \chi_2^2 \Pi_\alpha^2 R(\overline{z})\chi_1\varphi\rangle. \end{aligned} \tag{A.9}$$

Using that $(\Pi_\alpha \chi_2^2) = (2(p_y\chi_2) - Bx\chi_2)\chi_2 = \tilde{\chi}_2(2(p_y\chi_2) - Bx\chi_2)\chi_2$, and that $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$, we have

$$\begin{aligned} &|\langle R(\overline{z})\chi_1\varphi, (\Pi_\alpha \chi_2^2)\Pi_\alpha R(\overline{z})\chi_1\varphi\rangle| \\ &\qquad \leq \|2(p_y\chi_2) - Bx\chi_2\|_\infty \|\tilde{\chi}_2 R(\overline{z})\chi_1\varphi\| \|\chi_2\Pi_\alpha R(\overline{z})\chi_1\varphi\| \\ &\qquad \leq \frac{1}{2}\|2(p_y\chi_2) - Bx\chi_2\|_\infty^2 \|\tilde{\chi}_2 R(\overline{z})\chi_1\varphi\|^2 + \frac{1}{2}\|\chi_2\Pi_\alpha R(\overline{z})\chi_1\varphi\|^2 \end{aligned} \tag{A.10}$$

Combining (A.9) and (A.10) with $\alpha = x, y$, yields

$$\begin{aligned} &\frac{1}{2}\|\chi_2\Pi_x R(\overline{z})\chi_1\varphi\|^2 + \frac{1}{2}\|\chi_2\Pi_y R(\overline{z})\chi_1\varphi\|^2 \\ &\qquad \leq \frac{1}{2}(4\|p_x\chi_2\|_\infty^2 + (2\|p_y\chi_2\|_\infty + \|Bx\chi_2\|_\infty)^2)\|\tilde{\chi}_2 R(\overline{z})\chi_1\varphi\|^2 \\ &\qquad\quad + \left|\langle R(\overline{z})\chi_1\varphi, \chi_2^2(\Pi_x^2 + \Pi_y^2)R(\overline{z})\chi_1\varphi\rangle\right| \\ &\qquad \leq \frac{1}{2}(4\|p_x\chi_2\|_\infty^2 + 8\|p_y\chi_2\|_\infty^2 + 2\|Bx\chi_2\|_\infty^2)\|\tilde{\chi}_2 R(\overline{z})\chi_1\varphi\|^2 \\ &\qquad\quad + (|z| + \|V\|_\infty)\|\chi_2 R(\overline{z})\chi_1\varphi\|^2 + \|\chi_2\chi_1\varphi\|\|\chi_2 R(\overline{z})\chi_1\varphi\|. \end{aligned}$$

Inequality (A.6) follows. As for (A.7), notice that $[H, \mathcal{X}] = -2i\Pi_y\mathcal{X}' - \mathcal{X}'' = (-2i\Pi_y\mathcal{X}' - \mathcal{X}'')\tilde{\mathcal{X}}$. We thus apply (A.6) with $(\mathcal{X}'\chi_2)$ in place of χ_2. The lemma follows. □

B. Appendix B: Some trace estimates

Lemma 4. *Let V be a bounded potential, and denote by R_1 and R_2 the resolvents of operators H_1 and H_2 as in (2.1). Set $\eta_i = \mathrm{dist}(z, \sigma(H_i))$, $i = 1, 2$. There exists $C_1, C_2 > 0$ such that for any $(x, y) \in \mathbb{R}^2$, and H_1 and H_2 s.t. $\|V_1 - V_2\|_\infty < \infty$,*

$$\|R_1(z)R_2(z)V\mathbf{1}_{(x,y)}\|_1 \leq \frac{C_1\|V\mathbf{1}_{(x,y)}\|_\infty}{\eta_1\eta_2}\,(1 + C_2\|(V_1 - V_2)\|_\infty)\,, \tag{B.1}$$

and

$$\|R_1(z)V\mathbf{1}_{(x,y)}R_2(z)\|_1 \le \frac{C_1\|V\mathbf{1}_{(x,y)}\|_\infty}{\eta_1\eta_2}\left(1 + C_2\|(V_1 - V_2)\|_\infty\right). \qquad \text{(B.2)}$$

Proof. First note that setting $\overline{\chi}_{(x,y)} = V\mathbf{1}_{(x,y)}/\|V\mathbf{1}_{(x,y)}\|_\infty$, it is enough to bound $\|R_1(z)R_2(z)\overline{\chi}_{(x,y)}\|_1$ and $\|R_1(z)\overline{\chi}_{(x,y)}R_2(z)\|_1$ with $|\overline{\chi}_{(x,y)}| \le 1$ and supported on the unit cube centered at (x,y). Now choose $M \in \mathbb{R}$ below the spectrum of H_1 and H_2.

We first prove (B.2). By the resolvent identity,

$$\begin{aligned}\|R_1(z)\overline{\chi}_{(x,y)}R_2(z)\|_1 &\le \frac{C(M)}{\eta_1\eta_2}\|R_1(M)\overline{\chi}_{(x,y)}R_2(M)\|_1\\ &\le \frac{C(M)}{\eta_1\eta_2}\|R_1(M)\overline{\chi}_{(x,y)}R_1(M)\|_1(1+\|(V_2-V_1)R_2(M)\|)\\ &\le \frac{C(M)}{\eta_1\eta_2}\|R_1(M)|\overline{\chi}_{(x,y)}|R_1(M)\|_1(1+\|(V_2-V_1)R_2(M)\|).\end{aligned}$$

And (B.2) follows since $\|R_1(M)|\overline{\chi}_{(x,y)}|R_1(M)\|_1 = \|R_1(M)\sqrt{|\overline{\chi}_{(x,y)}|}\|_2 < C$ uniformly in (x,y), e.g. [Si] [GK2, Lemma A.4]. We turn to (B.1). By the resolvent identity,

$$\begin{aligned}\|R_1(z)R_2(z)\overline{\chi}_{(x,y)}\|_1 &\le \frac{C(M)}{\eta_1}\|R_1(M)R_2(z)\overline{\chi}_{(x,y)}\|_1\\ &\le \frac{C(M)}{\eta_1}\|R_2(M)R_2(z)\overline{\chi}_{(x,y)}\|_1(1+\|R_1(M)(V_2-V_1)\|)\\ &\le \frac{C(M)}{\eta_1\eta_2}\|R_2(M)^2\overline{\chi}_{(x,y)}\|_1(1+\|R_1(M)(V_2-V_1)\|).\end{aligned}$$

And (B.1) follows since $\|R_2(M)^2\overline{\chi}_{(x,y)}\|_1 < C$ uniformly in (x,y), e.g. [Si][GK2, Lemma A.4]. □

Lemma 5. *Suppose $I = [a,b] \subset]B_N, B_{N+1}[$, and pick a switch function g s.t. $\operatorname{supp} g' \subset I$. Suppose that $V_1(x,y) > b$ if $x < -R$ for some $R > 0$ (i.e. V_1 is a left confining potential). Then $g'(H(V_1,0))i[H_L,\mathcal{X}]$ is trace class.*

Proof. Technical details are similar to the ones used to prove Theorem 1 and Theorem 2. We thus only sketch the main ideas. We split $g'(H(V_1,0))i[H_L,\mathcal{X}]$ in two terms: $g'(H(V_1,0))\mathbf{1}_{x<-R}i[H_L,\mathcal{X}]$ and $g'(H(V_1,0))\mathbf{1}_{x\ge -R}i[H_L,\mathcal{X}]$. Let $\beta = B_{N+1} + \inf(V_1\mathbf{1}_-)$. Note that $g'(H(V_1,0)+\beta\mathbf{1}_{x>-R}) = 0$ for I does not intersect the spectrum of $H(V_1,0)+\beta\mathbf{1}_{x>-R}$ (which starts above b). The first term can thus be seen to be trace class by decomposing $\{g'(H(V_1,0)) - g'(H(V_1,0)+\beta\mathbf{1}_{x>-R})\}\mathbf{1}_{x<-R}i[H_L,\mathcal{X}]$ with the Helffer-Söjstrand formula and using the resolvent identity in the spirit of the proof of Theorem 1 and Theorem 2. The second term is seen to be trace class by noting that $g'(H_L) = 0$ and by considering $\{g'(H(V_1,0)) - g'(H_L)\}\mathbf{1}_{x>-R}i[H_L,\mathcal{X}]$ in the same way (taking advantage of $H(V_1,0) - H_L = V_1\mathbf{1}_-$). □

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