GENERALIZED FRACTAL DIMENSIONS ON THE NEGATIVE AXIS FOR NON COMPACTLY SUPPORTED MEASURES

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ABSTRACT. We study the finiteness of the generalized fractal dimensions $D^{\pm}_{\mu}(q)$ (also called Hentschel-Procaccia dimensions) for a non compactly supported measure μ on a complete metric space, and for $q < 0$. The upper dimensions are shown to be always infinite. We then provide a sufficient condition for the lower dimensions to bemeasures infinite. Optimality of our theorems is proved by constructing explicit measures on $\mathbb{R}$.

1. Introduction and results

Let (X, ϱ) be a complete metric space, and μ a positive Borel measure on X with finite total mass: $\mu(X) < \infty$. We denote by $B(x, \varepsilon)$ the ball centered at x and of radius ε, *i.e.* $B(x, \varepsilon) = \{y \in X, \varrho(x, y) \leq \varepsilon\}$. We denote by $\text{supp}\mu$ the support of μ, that is the smallest closed set F such that $\mu(X \backslash F) = 0$ [M]. It is well-known (e.g. [M]) that for Borel measures on a separable metric space, $\text{supp}\mu$ is a well defined (unique) set and one has

$$\text{supp}\mu = \{x \in X,\ \mu(B(x, \varepsilon)) > 0 \text{ for any } \varepsilon > 0\}. \tag{1.1}$$

Note that with our assumptions on X and μ, one has $\mu(X \backslash \text{supp}\mu) = 0$. Recall that the functions $x \to \mu(B(x, \varepsilon))$ are μ-measurable. The space X is said to be unbounded with repsect to ϱ if for any $R > 0$, there exists points $x, y \in X$, s.t. $\varrho(x, y) \geq R$, and μ is said to have an unbounded support if one can pick such x, y in $\text{supp}\mu$.

For $q \in \mathbb{R}$ and $\varepsilon \in (0, 1)$, we consider the following integrals with values in $\mathbb{R} \cup \{+\infty\}$:

$$I_{\mu}(q, \varepsilon) = \int_{\text{supp}\mu} \mu(B(x, \varepsilon))^{q-1} \text{d}\mu(x). \tag{1.2}$$

We define the following functions on $\mathbb{R}$ with values in $\mathbb{R} \cup \{+\infty\}$:

$$\tau^{-}_{\mu}(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log I_{\mu}(q, \varepsilon)}{\log(1/\varepsilon)}\ , \quad \tau^{+}_{\mu}(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log I_{\mu}(q, \varepsilon)}{\log(1/\varepsilon)}\ , \tag{1.3}$$

with the understanding that $\tau^{\pm}_{\mu}(q) = +\infty$ if for some $\varepsilon > 0$, $I_{\mu}(q, \varepsilon) = +\infty$. The *Hentschel-Procaccia dimensions* are given by

(1.4)

$$D^{-}_{\mu}(q) = \liminf_{\varepsilon \downarrow 0} \frac{\log I_{\mu}(q, \varepsilon)}{(1 - q) \log(1/\varepsilon)}\ , \quad D^{+}_{\mu}(q) = \limsup_{\varepsilon \downarrow 0} \frac{\log I_{\mu}(q, \varepsilon)}{(1 - q) \log(1/\varepsilon)}\ , \quad q \neq 1,$$

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with values in $[0, +\infty]$. We recall that these dimenions are non increasing functions of $q \neq 1$.

These dimensions have discrete analogs, which are called generalized Rényi dimensions, and where the integral $I_\mu(q, \varepsilon)$ is replaced by the sum

$$S_\mu(u, q, \varepsilon) = \sum_{i | B(x_i, \varepsilon) \in u} \mu(B(x_i, \varepsilon)^q,$$

with $u = (B(x_i, \varepsilon))_{i \in I}$ a collection of closed balls of radius ε and center $x_i \in X$. Then one considers either the infimum of $S_\mu(u, q, \varepsilon)$ over all possible (centered) *coverings* u of suppμ, or the supremum of $S(u, q, \varepsilon)$ over all possible (centered) *packings* u of suppμ. This leads respectively to the dimensions $P_c D_\mu^\pm(q)$ and $C_c D_\mu^\pm(q)$. We refer to [GT] for a precise description and basic properties. We note that if $q < 0$ then in great generality $P_c D_\mu^+(q) = C_c D_\mu^+(q)$.

Hentschel-Procaccia and Rényi dimensions turn out to be useful in a wide range of domains. They either describe various physical phenomena [Be, BGT1, GP, GKT, HLMV, HJKPS, HP, Ma, TV1, TV2] or are used to study statistical data [Be, Ha, VBP].

Finiteness as well as equality of Hentschel-Procaccia and generalized Rényi dimensions are basic issues studied in numerous works [Cu, O, P, LN, GY, BGT2, GT]. Concerning the equality of these dimensions, one can identify three different regimes: $q > 1$, $q \in]0, 1[$, and $q < 0$. These regimes have to be studied separately for they discriminate between different statistical properties of the measure μ. The first two regimes are by now well understood [O, P, BGT2].

The understanding of the validity of the equality $D_\mu^\pm(q) = P_c D_\mu^\pm(q)$ in the regime $q \leq 0$ for *compactly* supported measures is the main contribution of [GT]. It is proved there that $D_\mu^+(q)$ and $P_c D_\mu^+(q)$, $q < 0$, always coincide, being finite or not; moreover if they are finite then the lower dimensions coincide as well. If the upper dimensions are infinite, then lower dimensions $D_\mu^-(q)$ and $P_c D_\mu^-(q)$ may or may not coincide and we provide criteria for that.

In this article we prove similar results but for *non compactly* supported measures. In this case, it is quite immediate to see that $P_c D_\mu^\pm(q) = +\infty$ [LN, GT]. We shall first prove that the upper dimensions always coincide. We indeed have the

Theorem 1. *Suppose μ has a non compact support. Then for any $q < 0$,*

$$\limsup_{\varepsilon \downarrow 0} \frac{\log \log \log I_\mu(q, \varepsilon)}{\log(1/\varepsilon)} \geq 1. \tag{1.5}$$

In particular for any $q < 0$, $D_\mu^+(q) = +\infty$.

Theorem 1 is optimal as shown in Section 4. We exhibit a family measures on $\mathbb{R}$ with the limit in (1.5) arbitrary close to 1, showing that the bound (1.5) cannot be improved.

We then turn to the lower dimensions and we further assume that the support of the measure μ is unbounded with respect to the metric ϱ on X. In the particular case of $X = \mathbb{R}^d$, endowed with the usual euclidian distance, unboundedness is equivalent to non compactness. As shown in Section 4 the dimensions $D_\mu^-(q)$ need not to be infinite, but we prove the following optimal criteria:

Theorem 2. *Suppose* suppμ *is unbounded. If*

$$\limsup_{r\to 0} \frac{\log\log\log\log F(r)^{-1}}{r} < +\infty, \text{ where } F(r) = \frac{\mu(X\backslash B(0,r))}{\mu(X)} \in]0,1[, \tag{1.6}$$

then $D_\mu^-(q) = +\infty$ *for any* $q < 0$.

2. Proof of Theorem 1

By [GT, Lemma 2.1], there exists $\tilde{\varepsilon} > 0$ so that for any $\varepsilon < \tilde{\varepsilon}$ there exists a centered packing family $(B(x_i,\varepsilon))_{i\in\mathbb{N}}$ of suppμ which has infinite cardinality. Note that for any $\varepsilon' \leq \varepsilon$, the family $(B(x_i,\varepsilon'))_{i\in\mathbb{N}}$ is still a centered packing (with infinite cardinality). So we assume that this packing does not cover suppμ totally.
Suppose the theorem does not hold: there exists $q < 0$ such that (1.5) fails, meaning that there exists $\alpha \in (0,1)$, and some $\varepsilon^* > 0$ such that for all $\varepsilon < \varepsilon^*$, one has

$$I_\mu(q,\varepsilon) \leq \exp\left(\exp \varepsilon^{-\alpha}\right). \tag{2.1}$$

We define $B > 0$ by

$$B_0\left(\frac{1}{\alpha} - 1\right) = 2 \text{ and } B = \max(1, B_0). \tag{2.2}$$

Given the constant $K_{q,\alpha} > 0$ coming from (2.8), we fix $\eta_0 < \min(1, K_{q,\alpha}^{-1}, \tilde{\varepsilon}, \varepsilon^*)$. For any $\varepsilon \in (0,\eta_0)$, set

$$F_n(\varepsilon) = \frac{1}{\mu(X)}\sum_{i\geq n}\mu(B(x_i,\varepsilon)) \quad \text{and} \quad f_n(\varepsilon) = \log\frac{1}{F_n(\varepsilon)}.$$

So $F_n(\varepsilon) < 1$ and thus $f_n(\varepsilon) > 0$ for $\varepsilon \leq \eta_0$. The packing being centered, the sum in $F_n(\varepsilon)$, for any $\varepsilon \leq \eta_0$, contains infinitely many non zero terms, so that $F_n(\varepsilon) > 0$ for any n, and $f_n(\varepsilon) < +\infty$. Note also that $\lim_{n\to\infty} F_n(\eta_0) = 0$. To alleviate the notations we shall assume $\mu(X) = 1$.

For $\eta_0 > 0$ fixed (given above) we pick n large enough so that

$$\log\left(\frac{|q|}{2}f_n(\eta_0)\right) > \eta_0^{-B}. \tag{2.3}$$

This is possible since $\lim_{n\to\infty} F_n(\eta_0) = 0$. It thus fixes the function f_n that enters the play. Note that, for any $\varepsilon \in (0,\eta_0)$,

$$\begin{aligned} I_\mu(q,\varepsilon) &\geq \sum_{i\geq n}\int_{B(x_i,\eta_0-\varepsilon)} \mathrm{d}\mu(x)\mu(B(x,\varepsilon))^{-(1+|q|)} \\ &\geq \sum_{i\geq n}\int_{B(x_i,\eta_0-\varepsilon)} \mathrm{d}\mu(x)F_n(\eta_0)^{-(1+|q|)} = F_n(\eta_0-\varepsilon)F_n(\eta_0)^{-(1+|q|)}. \end{aligned} \tag{2.4}$$

By taking the log, (2.4) together with (2.1) leads to

$$f_n(\eta_0-\varepsilon) \geq (1+|q|)f_n(\eta_0) - \exp\left(\exp\varepsilon^{-\alpha}\right). \tag{2.5}$$

The latter holds whenever $\varepsilon \in (0,\eta_0)$. Given η_0, define the sequences $(\eta_k)_{k\in N}$ and $(\varepsilon_k)_{k\in N}$ by

$$\varepsilon_k = \left(\log\frac{|q|}{2}f_n(\eta_k)\right)^{-1/\alpha}, \quad \eta_{k+1} = \eta_k - \varepsilon_k. \tag{2.6}$$

It leads to $f_n(\eta_{k+1}) \geq (1+|q|/2)f_n(\eta_k)$ for any $k \geq 0$, and thus

$$f_n(\eta_k) > (1+|q|/2)^k f_n(\eta_0). \tag{2.7}$$

To prove the theorem we show that $\eta_\infty := \lim_{k\to\infty} \eta_k > 0$. Indeed this will imply $f_n(\eta_\infty) < \infty$, in contradiction with (2.7). First note that by definition of ε_k in (2.6) together with (2.3) and (2.7) one has

$$\varepsilon_k < \left(\log\left(\frac{|q|}{2}(1+|q|/2)^k f_n(\eta_0)\right)\right)^{-1/\alpha} < \left(k\log(1+|q|/2)+\eta_0^{-B}\right)^{-1/\alpha}.$$

Since $\alpha < 1$, $\sum_k \varepsilon_k < \infty$ and thus η_∞ exists and is finite. It remains to show that $\eta_\infty > 0$, *i.e.* that $\sum_k \varepsilon_k < \eta_0$:

$$\sum_{k=0}^{+\infty} \varepsilon_k < \sum_{k=0}^{+\infty} \left(k\log(1+|q|/2)+\eta_0^{-B}\right)^{-1/\alpha} \le K_{q,\alpha}\eta_0^{B(1/\alpha-1)} \le K_{q,\alpha}\eta_0^2. \tag{2.8}$$

The second majoration in (2.8) holds for some constant $K_{q,\alpha}$ that can be explicitely computed (for instance by comparing the sum to an integral), and to get the last bound we used (2.2). Since in addition we picked $\eta_0 < K_{q,\alpha}^{-1}$, it follows that $\eta_\infty > 0$.

3. Proof of Theorem 2

We denote by $F(r)$ and by $f(r)$ the functions

$$F(r) = \frac{\mu(X\backslash B(0,r))}{\mu(X)} \quad \text{and} \quad f(r) = \log F(r)^{-1}. \tag{3.1}$$

Note that since the support of μ is assumed to be unbounded, one has $F(r) \in]0,1[$ for any $r > 0$, and thus $f(r) > 0$ is finite. With no loss assume that $\mu(X) = 1$.

Suppose that (1.6) fails for some $q < 0$. Then there exists a finite constant $A > 0$, and some sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ such that for all $k \ge 0$,

$$I_\mu(q,\varepsilon_k) \le \varepsilon_k^{-A}. \tag{3.2}$$

Note that for any $r,\varepsilon > 0$ if $y \in X\backslash B(0,r+\varepsilon)$ then $B(y,\varepsilon) \subset X\backslash B(0,r)$. Thus

$$\begin{aligned} I_\mu(q,\varepsilon_k) &\ge \int_{X\backslash B(0,r+\varepsilon_k)} \mu(B(y,\varepsilon_k))^{q-1}\mathrm{d}\mu(y) \ge \int_{X\backslash B(0,r+\varepsilon_k)} F(r)^{q-1}\mathrm{d}\mu(y) \\ &\ge F^{q-1}(r)F(r+\varepsilon_k). \end{aligned} \tag{3.3}$$

Taking the log in (3.2)-(3.3) and recalling the definition of $f(r)$ in (3.1), one has, for any $r > 0$ and any $k \ge 0$,

$$f(r+\varepsilon_k) \ge (1+|q|)f(r) - A\log(1/\varepsilon_k). \tag{3.4}$$

Suppose now that for ε_k fixed, r is large enough so that $(|q|/2)f(r) \ge A\log(1/\varepsilon_k)$, then by a repeated use of (3.4), one has, for any $\nu \ge 0$,

$$f(r+\nu) \ge \left(1+\frac{|q|}{2}\right)^{\left[\frac{\nu}{\varepsilon_k}\right]} f(r), \tag{3.5}$$

where $[x]$ stands for the integer part of a real x. The key idea of the present proof is to pick ν large enough so that one can reach the next scale ε_{k+1}, by requiring $f(r+\nu) \ge A\log(1/\varepsilon_{k+1})$ (so that (3.4)-(3.5) holds for ε_{k+1}). If ε_k decays fast enough (see below) it will lead to the correct condition on the growth of $F(r)$.

By maybe extracting a new subsequence, we assume with no loss that the sequence $\varepsilon_k \le 1/\mathrm{e}$ and that the following fast decay holds: for any $k \ge 0$,

$$\log\log\varepsilon_{k+1}^{-1} \ge \varepsilon_k^{-1}\log\varepsilon_k^{-1}. \tag{3.6}$$

From (3.6) we get that for any $k \geq 1$,

$$\frac{\varepsilon_{k-1} \log\log \varepsilon_k^{-1}}{\varepsilon_k \log\log \varepsilon_{k+1}^{-1}} \leq \frac{1}{2}, \tag{3.7}$$

since, provided ε_1^{-1} is chosen large enough, one has

$$\frac{\log\log \varepsilon_k^{-1}}{\varepsilon_k \log\log \varepsilon_{k+1}^{-1}} \leq \frac{\log\log \varepsilon_k^{-1}}{\log \varepsilon_k^{-1}} \leq \varepsilon_0^{-1}/2 \leq \varepsilon_{k-1}^{-1}/2,$$

We now define two new sequences $(r_k)_{k\in\mathbb{N}}$ and $(\nu_k)_{k\in\mathbb{N}}$ as follows. Pick r_0 large enough so that

$$f(r_0) > \log(1+|q|/2) \max\left(1, \frac{2A}{|q|} \log \varepsilon_0^{-1}\right), \tag{3.8}$$

and for any $k \geq 0$,

$$\nu_k = \frac{\varepsilon_k \log\log \varepsilon_{k+1}^{-1}}{\log(1+|q|/2)}, \quad r_{k+1} = r_k + \nu_k. \tag{3.9}$$

It follows that for any $k \geq 0$,

$$\begin{aligned} f(r_{k+1}) = f(r_k+\nu_k) &\geq (1+|q|/2)^{[\nu_k/\varepsilon_k]} f(r_k) \\ &\geq (1+|q|/2)^{\nu_k/\varepsilon_k} \frac{2A}{|q|} \log \varepsilon_0^{-1} = \log \varepsilon_{k+1}^{-1} \frac{2A}{|q|} \log \varepsilon_0^{-1} \\ &\geq \frac{2A}{|q|} \log \varepsilon_{k+1}^{-1} \end{aligned} \tag{3.10}$$

(Recall $\log \varepsilon_0^{-1} \geq 1$). As a consequence using (3.10) in (3.4) leads to, for any $\nu > 0$ and any $k \geq 0$,

$$f(r_k+\nu) \geq \left(1+\frac{|q|}{2}\right)^{[\nu/\varepsilon_k]} f(r_k) \geq \left(1+\frac{|q|}{2}\right)^{\nu/\varepsilon_k}. \tag{3.11}$$

In order to conclude we first show that r_{k+1} is of the order of ν_k.

Lemma 3. *Recall* (3.6)-(3.7) *and* (3.9). *One has*
(i) $\lim_{k\to+\infty} r_k = +\infty$.
(ii) For any $k \geq 0$, $\nu_k \leq r_{k+1} \leq 2\nu_k + r_0$.

Proof. (i) By (3.7) $\nu_{j+1} \geq 2\nu_j$ and thus $r_k = r_0 + \sum_{j=0}^{k-1} \nu_j$ goes to infinity with k.
(ii) Obviously $\nu_k \leq r_{k+1}$. On the other hand, again by (3.7), for any $j = 0, 1, \cdots, k$, one has $\nu_j/\nu_k \leq 2^{j-k}$. Hence

$$r_{k+1} = r_0 + \nu_k \sum_{j=0}^{k} \nu_j/\nu_k \leq r_0 + \nu_k \sum_{j=0}^{k} 2^{j-k} \leq r_0 + 2\nu_k. \tag{3.12}$$

□

We are now in position to finish the proof of Theorem 2. Using (3.11) with $\nu = r_k$ leads to

$$\log f(2r_{k+1}) \geq \frac{r_{k+1}}{\varepsilon_{k+1}} \log(1+|q|/2) \geq \frac{1}{\varepsilon_{k+1}},$$

for k large enough. Thus, for any k large enough,

$$\begin{aligned}\log\log\log f(2r_{k+1}) &\geq \log\log\varepsilon_{k+1}^{-1} = \nu_k \log(1+|q|/2)\varepsilon_k^{-1} \\ &\geq \frac{1}{2}(r_{k+1}-r_0)\log(1+|q|/2)\varepsilon_k^{-1}\ ,\end{aligned}$$

by Lemma 3. Dividing by $2r_{k+1}$ and taking the limit k goes to infinity yields $\lim_{k\to+\infty}\log\log\log f(2r_{k+1})/2r_{k+1} = +\infty$, which contradicts Hypothesis (1.6).

4. Optimality

4.1. **Optimality of Theorem 1.** To see that Theorem 1 is optimal, we consider the measure $\mathrm{d}\mu_a(x) = \exp(-\exp(x^{1+a}))\mathrm{d}x$, with $a > 0$, on $[1,+\infty)$. Straightforward considerations will show that for any negative q, and $\nu > 0$, and for all ε small enough, depending on q, a and ν, one has

$$I_{\mu_a}(q,\varepsilon) \leq \exp\left(\exp\left(\frac{1}{\varepsilon}\right)^{1+\nu+\frac{1}{a}}\right), \tag{4.1}$$

which in turn implies that the limit in (1.5) is smaller than $1+\nu+\frac{1}{a}$. So if (4.1) holds then (1.5) is indeed the fastest growth in $1/\varepsilon$ of the integral $I_\mu(q,\varepsilon)$ that one can hope in full generality for a measure that is non compactly supported.

We prove (4.1). As a first step, we use that

$$\mu(x-\varepsilon,x+\varepsilon) \geq \mu(x-\varepsilon,x-\frac{\varepsilon}{2}) \geq \frac{\varepsilon}{2}\exp\left(-\exp\left(x-\frac{\varepsilon}{2}\right)^{1+a}\right). \tag{4.2}$$

This leads to

$$\begin{aligned}&\int_1^\infty \mu(x-\varepsilon,x+\varepsilon)^{-(1+|q|)}\mathrm{d}\mu(x) \leq \int_1^\infty \mu(x-\varepsilon,x-\frac{\varepsilon}{2})^{-(1+|q|)}\mathrm{d}\mu(x) \\ &\leq \left(\frac{\varepsilon}{2}\right)^{-(1+|q|)}\int_1^\infty \exp\left((1+|q|)\exp\left(x-\frac{\varepsilon}{2}\right)^{1+a} - \exp x^{1+a}\right)\mathrm{d}x.\end{aligned} \tag{4.3}$$

Let $\delta > 0$. We divide the half line in two pieces: $x \in [1,\varepsilon^{-\frac{1+\delta}{a}}]$, and $x \in]\varepsilon^{-\frac{1+\delta}{a}},\infty[$. In the first case we trivially bound $\exp(-\exp x^{1+a})$ by 1 and get

$$\begin{aligned}\int_1^{\varepsilon^{-\frac{1+\delta}{a}}}\frac{\mathrm{d}\mu(x)}{\mu(x-\varepsilon,x+\varepsilon)^{1+|q|}} &\leq \left(\frac{2}{\varepsilon}\right)^{1+|q|}\left(\frac{1}{\varepsilon}\right)^{\frac{1+\delta}{a}}\exp\left((1+|q|)\exp\left(\varepsilon^{-\frac{(1+\delta)(1+a)}{a}}\right)\right) \\ &\leq \exp\left(\exp\left(\varepsilon^{-\frac{(1+2\delta)(1+a)}{a}}\right)\right),\end{aligned} \tag{4.4}$$

for ε small enough depending on δ, a, and q. We turn to the case $x \geq \varepsilon^{-\frac{1+\delta}{a}}$. First note that for ε small enough (depending on a and δ), and uniformly in $x \geq \varepsilon^{-\frac{1+\delta}{a}}$

one has $(1-\frac{\varepsilon}{2x})^{1+a} \le 1 - \frac{1+a}{2}\frac{\varepsilon}{2x}$. It thus follows from (4.3) that

$$\int_{\varepsilon^{-\frac{1+\delta}{a}}}^{\infty} \frac{\mathrm{d}\mu(x)}{\mu(x-\varepsilon,x+\varepsilon)^{1+|q|}}$$

$$\le \left(\frac{2}{\varepsilon}\right)^{1+|q|} \int_{\varepsilon^{-\frac{1+\delta}{a}}}^{\infty} \exp\left((1+|q|)\exp\left(x^{1+a}\left(1-\frac{(1+a)\varepsilon}{4x}\right)\right) - \exp x^{1+a}\right)\mathrm{d}x$$

(4.5)

$$\le \left(\frac{2}{\varepsilon}\right)^{1+|q|} \int_{\varepsilon^{-\frac{1+\delta}{a}}}^{\infty} \exp\left(\exp x^{1+a}\left((1+|q|)\mathrm{e}^{-\frac{1+a}{4}x^a\varepsilon}-1\right)\right)\mathrm{d}x$$

(4.6)

$$\le \left(\frac{2}{\varepsilon}\right)^{1+|q|} \int_{1}^{\infty} \exp\left(-\frac{1}{2}\exp x^{1+a}\right)\mathrm{d}x \le C\left(\frac{2}{\varepsilon}\right)^{1+|q|},$$

where in (4.5) we use that if $x \ge \varepsilon^{-\frac{1+\delta}{a}}$ then $x^a\varepsilon \ge \varepsilon^{-\delta}$, and in the first inequality of (4.6) we took ε small enough depending on δ, a, and q. Putting together (4.4) and (4.6) leads to (4.1).

4.2. **Optimality of Theorem 2.** We construct a measure on $\mathbb{R}^+$ (with unbounded support) with finite total mass, which has finite lower Hentschel-Procaccia dimension $D^-_\mu(q)$ for any $q<0$.

For $\nu>0$ with define by induction a decreasing sequence $(\varepsilon_j)_{j\in\mathbb{N}}$:

$$\varepsilon_0 = 1, \quad \log\log\varepsilon_j^{-1} = \varepsilon_{j-1}^{-(1+\nu)}\ , \tag{4.7}$$

with $\lim_{j\to\infty}\varepsilon_j = 0$, and an increasing sequence $(\rho_j)_{j\in\mathbb{N}}$ by

$$\rho_j = \left(\log\log\varepsilon_j^{-1}\right)^{\delta} = \varepsilon_{j-1}^{-\delta(1+\nu)}\ , \tag{4.8}$$

$\lim_{j\to\infty}\rho_j = +\infty$, where δ is chosen small enough so that

$$\delta(1+\nu) \le \nu\ . \tag{4.9}$$

(This condition will be used in (4.16) below). We define a measure μ by

$$\begin{cases} \mathrm{d}\mu(x) = \exp\left(-\exp(x/\varepsilon_j)\right)\mathrm{d}x \text{ on } I_j \equiv [\rho_j+1, \rho_{j+1}-1) \\ \mathrm{d}\mu(x) = 0 \text{ elsewhere}\ . \end{cases} \tag{4.10}$$

The measure μ has obviously a finite mass. We claim that for any $q<0$ one has $D^-_\mu(q) < +\infty$. We actually show that

$$\tau^-_\mu(q) \le 1-2q+2\nu \quad\text{and}\quad D^-_\mu(q) \le \frac{1-2q+2\nu}{1-q}, \tag{4.11}$$

and thus $D^-_\mu(-\infty) \le 2$. Moreover we check that

$$\limsup_{r\to\infty} \frac{\log\log\log\log F(r)^{-1}}{r} = +\infty, \tag{4.12}$$

so that Theorem 2 is optimal.

We start by the following observation. Suppose that $x\in I_j$, so that $[x-\varepsilon, x+\varepsilon] \subset [\rho_j, \rho_{j+1})$. Then

$$\mu([x-\varepsilon,x+\varepsilon]) \ge \int_{x-\varepsilon}^{x-\varepsilon/2} \exp\left(-\exp(x/\varepsilon_j)\right)\mathrm{d}x$$

$$\ge \frac{\varepsilon}{2}\exp(-\exp((x-\varepsilon/2)/\varepsilon_j))\ . \tag{4.13}$$

To evaluate the integral $I_\mu(q,\varepsilon)$, for $q<0$, we cut it in integrals $I_{\mu,j}(q,\varepsilon)$ computed over intervals I_j. It follows from (4.13) that for any $j\geq 0$,

$$\begin{aligned} I_{\mu,j}(q,\varepsilon) &= \int_{I_j} \mu([x-\varepsilon,x+\varepsilon])^{-(1+|q|)} \exp\left(-\exp\frac{x}{\varepsilon_j}\right) \mathrm{d}x \\ &\leq \left(\frac{2}{\varepsilon}\right)^{1+|q|} \int_{\rho_j}^{\rho_{j+1}} \exp\left((1+|q|)\exp\frac{x-\varepsilon/2}{\varepsilon_j}\right) \exp\left(-\exp\frac{x}{\varepsilon_j}\right) \mathrm{d}x \\ &= \left(\frac{2}{\varepsilon}\right)^{1+|q|} \int_{\rho_j}^{\rho_{j+1}} \exp\left(b_j(\varepsilon)\exp\frac{x}{\varepsilon_j}\right) \mathrm{d}x, \qquad (4.14) \end{aligned}$$

where

$$b_j(\varepsilon) = -1 + (1+|q|)\exp\left(-\frac{\varepsilon}{2\varepsilon_j}\right) . \qquad (4.15)$$

We make the following observations.
OBSERVATION 1: for any $q<0$, $j\geq 0$, $\varepsilon>0$, $b_j(\varepsilon)\leq |q|$,
OBSERVATION 2: for any $q<0$, there exists a finite constant $A(q)>0$, so that for any $j\geq k\geq 0$, $b_j(A(q)\varepsilon_k)\leq -1/2$.

Observation 1 is immediate. As for Observation 2, note that for any $j\geq k$, since the sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ is decreasing,

$$b_j\left(A(q)\varepsilon_k\right) \leq -1 + (1+|q|)\exp\left(-A(q)/2\right).$$

For $A(q)$ large enough, depending on q, Observation 2 follows.

We now evaluate $I_\mu(q,\varepsilon)$ for the particular values of ε, namely, $\varepsilon = A(q)\varepsilon_k$, and show that $I_\mu(q,A(q)\varepsilon_k)$ does not grow faster than a power of $(1/\varepsilon_k)$, which is exactly our claim. To achieve that we estimate integrals $I_{\mu,j}(q,\varepsilon)$, $j\geq 0$. We already got the upper bound (4.14). Consider now (4.15). If $j<k$ then $\varepsilon_k/\varepsilon_j$ is very small and thus $b_j(A(q)\varepsilon_k)$ is close to $|q|$. We shall therefore use Observation 1 to bound I_j. On the other hand, if $j>k$ then $\varepsilon_k/\varepsilon_j$ is large and $b_j(A(q)\varepsilon_k)$ is close to -1. This latter case, together with the case $j=k$, is handled by Observation 2.

So, for any $j<k$, on the account of (4.14) and Observation 1, recalling (4.8),

$$\begin{aligned} I_{\mu,j}(q,A(q)\varepsilon_k) &\leq \left(\frac{2}{A(q)\varepsilon_k}\right)^{1+|q|} \int_{\rho_j}^{\rho_{j+1}} \exp\left(|q|\exp\frac{x}{\varepsilon_j}\right) \mathrm{d}x \\ &\leq \left(\frac{2}{A(q)\varepsilon_k}\right)^{1+|q|} \rho_{j+1}\exp\left(|q|\exp\frac{\rho_{j+1}}{\varepsilon_j}\right) \mathrm{d}x \\ &\leq \left(\frac{2}{A(q)\varepsilon_k}\right)^{1+|q|} \left(\frac{1}{\varepsilon_j}\right)^{\delta(1+\nu)} \exp\left(|q|\exp\varepsilon_j^{-\delta(1+\nu)-1}\right) \mathrm{d}x \\ &\leq \left(\frac{2}{A(q)\varepsilon_k}\right)^{1+|q|} \left(\frac{1}{\varepsilon_j}\right)^{\nu} \exp\left(|q|\exp\varepsilon_j^{-\nu-1}\right) \mathrm{d}x \qquad (4.16) \\ &\leq \left(\frac{2}{A(q)\varepsilon_k}\right)^{1+|q|} \left(\frac{1}{\varepsilon_j}\right)^{\nu} \left(\frac{1}{\varepsilon_{j+1}}\right)^{|q|} \qquad (4.17) \\ &\leq C(q,\nu)\left(\frac{1}{\varepsilon_k}\right)^{1+2|q|+\nu}, \qquad (4.18) \end{aligned}$$

where in (4.16) we used (4.9), in (4.17) the definition of the ε_j's given in (4.7), and in (4.18), the fact that if $j<k$ then $\varepsilon_j\geq\varepsilon_{j+1}\geq\varepsilon_k$.

Now for the remainder part where $j \geq k$, note that on the account of (4.14) and Observation 2 of (recalling $\varepsilon_j \leq 1$),

$$\begin{aligned}\sum_{j\geq k} I_{\mu,j}(q, A(q)\varepsilon_k) &\leq \left(\frac{2}{A(q)\varepsilon_k}\right)^{1+|q|} \int_{\rho_k}^{+\infty} \exp\left(-\frac{1}{2}\exp\frac{x}{\varepsilon_j}\right) \mathrm{d}x \\ &\leq \left(\frac{2}{A(q)\varepsilon_k}\right)^{1+|q|} \int_{0}^{+\infty} \exp\left(-\frac{1}{2}\exp x\right) \mathrm{d}x \\ &= C'(q)\left(\frac{1}{\varepsilon_k}\right)^{1+|q|}. \end{aligned} \tag{4.19}$$

It follows that

$$\begin{aligned} I_\mu(q, A(q)\varepsilon_k) &= \sum_{j\geq 0} I_{\mu,j}(q, A(q)\varepsilon_k) \leq k\, C(q,\nu)\varepsilon_k^{-(1+2|q|+\nu)} + C'(q)\varepsilon_k^{-(1+|q|)} \\ &\leq C''(q,\nu)\varepsilon_k^{-(1+2|q|+2\nu)}, \end{aligned}$$

where in the latter inequality we used that $k \leq \varepsilon_k^{-\nu}$, which follows easily from the fast decay of the ε_k's in (4.7). Thus (4.11) holds.

We now show that the measure μ satisfies to (4.12). Take $r > 0$ and pick j such that $r \in [\rho_j, \rho_{j+1})$. One has for j (and thus r) large enough,

$$F(r) = \int_r^{+\infty} \mathrm{d}\mu(x) \leq 2\int_r^{\rho_{j+1}} \mathrm{d}\mu(x) \leq (\rho_{j+1} - r)\exp\left(-\exp\frac{r}{\varepsilon_j}\right),$$

and thus

$$\log F(r)^{-1} \geq \exp\frac{r}{\varepsilon_j} + \log\rho_{j+1}^{-1} \geq \exp\frac{r}{2\varepsilon_j}, \tag{4.20}$$

If $r = \rho_j$, then $\log\log F(\rho_j)^{-1} \geq \rho_j/(2\varepsilon_j) \geq 1/(2\varepsilon_j)$. Recalling (4.8) and $\delta < 1$ by (4.9) it leads to

$$\frac{\log\log\log\log F(\rho_j)^{-1}}{\rho_j} \geq \frac{\log\log\varepsilon_j}{2\rho_j} = \frac{1}{2}\rho_j^{1/\delta - 1} \to +\infty.$$

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