

# LOCALIZATION FOR A CONTINUUM CANTOR-ANDERSON HAMILTONIAN

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*Dedicated to Jean-Michel Combes on the occasion of his 65<sup>th</sup> birthday.*

ABSTRACT. We prove localization at the bottom of the spectrum for a random Schrödinger operator in the continuum with a single-site potential probability distribution supported by a Cantor set of zero Lebesgue measure. This distribution is too singular to be treated by the usual methods. In particular, an “a priori” Wegner estimate is not available. To prove the result we perform a multiscale analysis following the work of Bourgain and Kenig for the Bernoulli-Anderson Hamiltonian, and obtain the required Wegner estimate scale by scale. To do so, we generalize their argument based on Sperner’s Lemma by resorting to the LYM inequality for multisets, and combine it with the concept of scale dependent equivalent classes of configurations introduced by Germinet, Hislop and Klein for the study of Poisson Hamiltonians.

## 1. INTRODUCTION AND SETTING

Consider the continuum Cantor-Anderson Hamiltonian

$$H_{\omega} := -\Delta + V_{\omega} \quad \text{on } L^2(\mathbb{R}^d), \tag{1.1}$$

where the potential is given by

$$V_{\omega}(x) := \sum_{\zeta \in \mathbb{Z}^d} \omega_{\zeta} u(x - \zeta), \tag{1.2}$$

where

- The single-site potential  $u$  is a nonnegative, nonzero  $L^{\infty}$ -function on  $\mathbb{R}^d$  with compact support, with

$$u_{-} \chi_{\Lambda_{\delta_{-}}(0)} \leq u \leq u_{+} \chi_{\Lambda_{\delta_{+}}(0)} \quad \text{for some constants } u_{\pm}, \delta_{\pm} \in ]0, \infty[, \tag{1.3}$$

$\Lambda_L(x)$  being the box of side  $L$  centered at  $x \in \mathbb{R}^d$ .

- $\omega = \{\omega_{\zeta}\}_{\zeta \in \mathbb{Z}^d}$  is a family of independent, identically distributed random variables, with a the common probability distribution  $\mu$  defined as the uniform measure on a Cantor set  $\mathcal{K} \subset [0, 1]$  of Lebesgue measure zero ( $\mu$  is constructed in Section 2). We take the underlying probability space to be  $(\Omega, \mathbb{P})$  with  $\Omega = \mathcal{K}^{\mathbb{Z}^d}$ ,  $\mathbb{P} = \mu^{\odot \mathbb{Z}^d}$ , and  $\{\omega_{\zeta}\}_{\zeta \in \mathbb{Z}^d}$  the coordinate functions.

Jean-Michel Combes has made major contributions to the study of continuum Anderson Hamiltonians, operators of the form (1.1)-(1.2) where the common probability distribution  $\mu$  is absolutely continuous with a bounded density. Combes and Hislop [CoH1] gave the first proof of Anderson localization for these random Schrödinger operators in the continuum. Combes and his collaborators [CoHM,

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A.K was supported in part by NSF Grant DMS-0457474.

CoHN, CoHKN, CoHK, CoHKR] made important contributions in the understanding of the Wegner estimate, culminating in the recent beautiful paper [CoHK], which contains the optimal a priori Wegner estimate, proving the long-sought Lipschitz continuity of the integrated density of states. Combes also made important contributions to the study of random Landau Hamiltonians [CoH2, CoHKR], which played an important role in the dynamical delocalization result obtained for this model in [GKS]. It is the pleasure of the authors to dedicate this paper to Jean-Michel Combes. This paper also deals with a Wegner estimate, but because of the singularity of the probability distribution, it requires alternative arguments than the ones developed by Jean-Michel Combes and his collaborators.

We now state our result. We use  $\chi_x$  to denote the characteristic function of the unit cube  $\Lambda_1(x)$ .

**Theorem 1.1.** *There exists  $E_0 > 0$  such that  $H_\omega$  exhibits Anderson localization as well as dynamical localization in the energy interval  $[0, E_0]$ . More precisely:*

- (Anderson localization) *There exists  $m > 0$  such that, with probability one the operator  $H_\omega$  has pure point spectrum in  $[0, E_0]$  with exponentially localized eigenfunctions with rate of decay  $m$ , i.e., if  $\phi$  is an eigenfunction of  $H_\omega$  with eigenvalue  $E \in [0, E_0]$  we have*

$$\|\chi_x \phi\| \leq C_{\omega, \phi} e^{-m|x|}, \quad \text{for all } x \in \mathbb{R}^d. \quad (1.4)$$

- (Dynamical localization) *For all  $s < \frac{3}{8}d -$  we have*

$$\mathbb{E} \left\{ \sup_{t \in \mathbb{R}} \left\| \langle x \rangle^{\frac{m}{2}} e^{-itH_\omega} \chi_{[0, E_0]}(H_\omega) \chi_0 \right\|_2^{\frac{2s}{m}} \right\} < \infty \quad \text{for all } m \geq 1. \quad (1.5)$$

We thus obtain Anderson and dynamical localization with a single-site potential probability measure that is purely singular continuous with respect to Lebesgue measure. As shown in Section 2,  $\mu$  is log log-Hölder continuous which is too singular to enable us to use standard results for the Wegner estimate (such a priori Wegner estimates can be used to prove localization for single-site potential probability measures which are at least log-Hölder continuous). As a consequence, the standard multiscale scale analysis [FrS, DrK, FK, GK1] cannot be exploited either. This is why the strong form of dynamical localization obtained in [GK1] is replaced by the somewhat weaker form (1.5) proved in [GK2].

To prove the result we perform a multiscale analysis following the work of Bourgain and Kenig for the Bernoulli-Anderson Hamiltonian [BoK], and obtain the required Wegner estimate scale by scale. To do so, we generalize the argument based on Sperner's Lemma used in [BoK] by resorting to the LYM inequality for multisets (e.g., [A]), and combine it with the concept of scale dependent equivalent classes of configurations introduced by Germinet, Hislop and Klein for the study of Poisson Hamiltonians [GHK1, GHK2].

Theorem 1.1 can be proven in great generality: the only requirement on the single-site potential probability measure  $\mu$  is that  $\{0, \tau\} \in \text{supp } \mu \subset [0, \tau]$  for some  $\tau > 0$ . The present note may be considered as an illustrative introduction to the general result proved in [GK2] using the concentration bound of [AGKW]. Indeed, where here we restrict ourselves to a uniform measure and use the explicit hierarchical structure of its support (a particular Cantor set), new arguments had to be developed in [AGKW, GK2] in order to treat arbitrary measures. In particular, a Bernoulli decomposition of random variables is developed in [AGKW], which

yields the concentration bound that extends the probabilistic consequences of the combinatorial Sperner's Lemma to general random variables. This Bernoulli decomposition, combined with the extension in [GK2] of the Bourgain-Kenig multiscale analysis to more general Bernoulli-Anderson Hamiltonians, which incorporate an additional background potential and for which the variances of the Bernoulli terms are uniformly positive, but not necessarily the same, yields pure point spectrum with probability one at the bottom of the spectrum in the general case [AGKW]. The full result, as stated in Theorem 1.1, with Anderson localization (pure point spectrum plus uniform exponential decay of eigenfunctions) and dynamical localization, is proved in [GK2] by an extension of the Bourgain-Kenig analysis.

## 2. CONSTRUCTION OF THE PROBABILITY MEASURE

We construct a Cantor set as follows. We fix  $\beta > 0$  and an initial scale  $L_1 > 1$ , define a sequence of scales  $L_{k+1} = L_k^\beta$ ,  $k = 1, 2, \dots$ , and set  $\alpha_k = \exp(-L_k)$ . Let  $\mathcal{K}^{(0)} = [0, 1]$ . We remove the middle part of the interval and keep at the edges two intervals:  $I_0^{(1)}$  on the left and  $I_1^{(1)}$  on the right, both of length  $\alpha_1$ . We then repeat the procedure of each of the intervals  $I_0^{(1)}$  and  $I_1^{(1)}$ . At generation  $k$ ,  $\mathcal{K}^{(k)}$  is the union of  $2^k$  disjoint intervals  $I_j^{(k)}$ ,  $j = (j_1, \dots, j_k) \in \{0, 1\}^k$ , each of them with length  $\alpha_k$ , namely

$$\mathcal{K}^{(k)} = \bigcup_{j \in \{0, 1\}^k} I_j^{(k)} \text{ with } |I_j^{(k)}| = \alpha_k = \exp(-L_k). \quad (2.1)$$

Note that two intervals  $I_j^{(k)}$  and  $I_{j'}^{(k)}$ ,  $j \neq j'$ , are separated by a gap of size at least

$$G_k := \alpha_{k-1} - 2\alpha_k \approx \exp(-L_{k-1}). \quad (2.2)$$

The Lebesgue measure of  $\mathcal{K}^{(k)}$  is given by

$$|\mathcal{K}^{(k)}| = 2^k \alpha_k = 2^k \exp(-L_1^{\beta^k}), \quad (2.3)$$

which goes to zero as  $k$  goes to infinity. We define

$$\mathcal{K} = \bigcap_{k=0}^{\infty} \mathcal{K}^{(k)}. \quad (2.4)$$

The set  $\mathcal{K}$  is a Cantor set of zero Lebesgue measure.

We construct the uniform measure  $\mu$  on the Cantor set  $\mathcal{K}$  as follows. At generation  $k$  we consider the uniform measure  $\mu^{(k)}$  on  $\mathcal{K}^{(k)}$ , i.e.,  $\mu^{(k)}$  is the normalized Lebesgue measure on each interval  $I_j^{(k)}$ , with  $\mu^{(k)}(I_j^{(k)}) = 2^{-k}$  with for all  $j$ . Note  $\mu^{(k)}([0, 1]) = 1$  by construction. The Cantor measure  $\mu$  is the unique weak limit of the  $\mu^{(k)}$ 's (to see uniqueness, it is enough to compute the limit on arbitrary intervals). The support of  $\mu$  is the Cantor set  $\mathcal{K}$ , which has zero Lebesgue measure by construction. The measure  $\mu$  is purely singular continuous.

It is actually easy to see that  $\mu$  is log log-Hölder continuous, but not better. Indeed for any  $k, j$  we have

$$\mu(I_j^{(k)}) = \mu^{(k)}(I_j^{(k)}) = 2^{-k}, \quad (2.5)$$

that is, with  $\varepsilon = |I_j^{(k)}| = \exp(-L_1^{\beta^k})$ ,

$$\mu(I_j^{(k)}) = \left( \frac{\log |\log \varepsilon|}{\log L_1} \right)^{-\log 2 / \log \beta} ;$$

on the other hand, for any interval  $I$  of size  $\varepsilon > 0$ , recalling (2.2), if  $G_{k+1} \leq \varepsilon < G_k$ , then  $I$  covers at most one interval  $I_j^{(k)}$ , and thus

$$\mu(I) \leq 2^{-k} \lesssim \left( \frac{\log |\log \varepsilon|}{\log L_1} \right)^{-\log 2 / \log \beta} .$$

### 3. ELEMENTS OF THE MULTISCALE ANALYSIS

Finite volume operators are defined as in [GHK2]. Given a box  $\Lambda = \Lambda_L(x)$  in  $\mathbb{R}^d$  and a configuration  $\omega \in [0, 1]^{\mathbb{Z}^d}$ , we set

$$H_{\omega, \Lambda} := -\Delta_\Lambda + V_{\omega, \Lambda} \quad \text{on} \quad L^2(\Lambda), \quad (3.1)$$

where  $\Delta_\Lambda$  is the Laplacian on  $\Lambda$  with Dirichlet boundary condition, and

$$V_{\omega, \Lambda} := \chi_\Lambda \sum_{\zeta \in \mathbb{Z}^d \cap \Lambda} \omega_\zeta u(x - \zeta). \quad (3.2)$$

The finite volume resolvent is  $R_{X, \Lambda}(z) := (H_{X, \Lambda} - z)^{-1}$ .

We will identify  $L^2(\Lambda)$  with  $\chi_\Lambda L^2(\mathbb{R}^d)$ . Note that in general we do not have  $V_{\omega, \Lambda} = \chi_\Lambda V_{\omega, \Lambda'}$  for  $\Lambda \subset \Lambda'$ , where  $\Lambda'$  may be a finite box or  $\mathbb{R}^d$ . But we always have

$$\chi_{\hat{\Lambda}} V_{\omega, \Lambda} = \chi_{\hat{\Lambda}} V_{\omega, \Lambda'}, \quad (3.3)$$

where

$$\hat{\Lambda} = \hat{\Lambda}_L(x) := \Lambda_{L - \delta_+}(x) \quad \text{with} \quad \delta_+ \text{ as in (1.3)}, \quad (3.4)$$

which suffices for the multiscale analysis.

The usual definition of “good” boxes for the multiscale analysis is as follows.

**Definition 3.1.** *Let  $\varepsilon \in ]0, 1[$  (small). Consider  $\omega \in [0, 1]^{\mathbb{Z}^d}$ , an energy  $E \in \mathbb{R}$ , and a rate of decay  $m > 0$ . A box  $\Lambda_L$  is said to be  $(\omega, E, m)$ -good if*

$$\|R_{\omega, \Lambda_L}(E)\| \leq e^{L^{1-\varepsilon}} \quad (3.5)$$

and

$$\|\chi_x R_{\omega, \Lambda_L}(E) \chi_y\| \leq e^{-m|x-y|}, \quad \text{for all } x, y \in \Lambda_L \text{ with } |x-y| \geq \frac{L}{10}. \quad (3.6)$$

But *goodness* of boxes does not suffice for the induction step in the multiscale analysis given in [Bo, BoK], which also needs an adequate supply of *free sites* to obtain a Wegner estimate at each scale. Given  $S \subset \Lambda \cap \mathbb{Z}^d$  and  $t_S = \{t_\zeta\}_{\zeta \in S} \in [0, 1]^S$ , we set

$$H_{\omega, t_S, \Lambda} := -\Delta_\Lambda + \chi_\Lambda V_{\omega_\Lambda, t_S} \quad \text{on} \quad L^2(\Lambda), \quad (3.7)$$

where

$$V_{\omega_\Lambda, t_S}(x) := \sum_{\zeta \in (\Lambda \cap \mathbb{Z}^d) \setminus S} \omega_\zeta u(x - \zeta) + \sum_{\zeta \in S} t_\zeta u(x - \zeta). \quad (3.8)$$

$R_{\omega, t_S, \Lambda}(z)$  will denote the corresponding finite volume resolvent. Following [BoK], sites belonging to  $S$  are called *free sites*, and variables  $t_\zeta$  with  $\zeta \in S$  are called *free variables*.

**Definition 3.2.** Consider  $\omega \in [0, 1]^{\mathbb{Z}^d}$ , an energy  $E \in \mathbb{R}$ , a rate of decay  $m > 0$ , and  $S \subset \widehat{\Lambda}$ . A box  $\Lambda_L$  is said to be  $(\omega, S, E, m)$ -good if we have (3.5) and (3.6) with  $R_{\omega, t_S, \Lambda}(E)$  for all  $t_S \in [0, 1]^S$ . In this case  $S$  consists of  $(\omega, E)$ -free sites for the box  $\Lambda_L$ .

Recall the sequence of scales  $L_k$  from the construction of the Cantor set  $\mathcal{K}$ . Let  $\Lambda = \Lambda_L$  be a cube of side  $L$ .

For  $\varepsilon > 0$  (the one in Definition 3.1), we define  $\varepsilon' > 0$  such that

$$(1 + \varepsilon')(1 - \varepsilon) = 1. \quad (3.9)$$

**Definition 3.3.** Given a cube  $\Lambda = \Lambda_L$ , two configurations  $\omega, \omega' \in \Omega$  are said to be equivalent at scale  $L \in ]L_{k-1}^{1+\varepsilon'}, L_k^{1+\varepsilon'}]$  (thus given  $L$ , the integer  $k$  is unique), if for all  $i \in \mathbb{Z}^d \cap \Lambda$ ,  $\omega_i$  and  $\omega'_i$  belong to the same interval  $I_j^{(k)}$  of  $\mathcal{K}^{(k)}$ , and we write  $\omega_i \sim_L \omega'_i$ ,  $i \in \mathbb{Z}^d \cap \Lambda$ , and  $\omega \sim_\Lambda \omega'$  for the full configuration. We denote by  $[\omega_i]_L$  the equivalent class of  $\omega_i$ , and by  $[\omega]_\Lambda$  that of  $\omega$  in  $\Lambda$ .

**Remark 3.4.** In other terms, given  $L$  as in Definition 3.3, the quotient of  $\mathcal{K} = \text{supp } \mu$  by the relation of equivalence  $\sim_L$  can be identified with the set of the intervals of  $\mathcal{K}^{(k)}$ , and hence with  $\{1, \dots, K\}$ , where  $K = 2^k$  is the number of such intervals. We also identify the set of all equivalence classes  $[\omega]_\Lambda$  with  $\{1, \dots, K\}^{\mathbb{Z}^d \cap \Lambda}$ , and by  $[\omega_i]_L = l$ ,  $1 \leq l \leq K$ , we mean that we identify  $[\omega_i]_L$  with the  $l^{\text{th}}$  interval of  $\mathcal{K}^{(k)}$ .

Remark 3.4 motivates the following definition.

**Definition 3.5.** Let  $\Lambda = \Lambda_{L_n}$ ,  $K = 2^n$ . We denote by  $C_\Lambda \simeq \{1, \dots, K\}^{\mathbb{Z}^d \cap \Lambda}$  the set of collections of intervals  $I_j^{(n)}$ , indexed by lattice points in  $\Lambda$ . An element of  $C_\Lambda$  will be denoted by  $[\Lambda]_\Lambda$  or just  $[\Lambda]$ .

Similarly, if  $A \subset \Lambda \cap \mathbb{Z}^d$ , then  $C_A \simeq \{1, \dots, K\}^A$  denotes the set collections of intervals  $I_j^{(n)}$ , indexed by points in  $A$ . An element of  $C_A$  will be denoted by  $[A]_\Lambda$  or just  $[A]$ .

When we want to stress that we work at resolution  $k$ , we write  $C_\Lambda^{(k)}$  and  $C_A^{(k)}$ .

We now define the basic events (“bevents”) that we shall use for the multiscale analysis. They correspond to [BoK]’s cylinders in our particular setting. We rely on the construction introduced in [GHK2] but the situation is a bit simpler since we do not have to introduce “acceptable” configurations as in [GHK2]. In some sense, all our configurations are “acceptable”.

**Definition 3.6.** Let us give  $\varepsilon > 0$  and a box  $\Lambda = \Lambda_L(x)$ , and  $B, S$  a partition of  $\Lambda \cap \mathbb{Z}^d$ . For  $k$  s.t.  $L \in ]L_{k-1}^{1+\varepsilon'}, L_k^{1+\varepsilon'}]$  we write  $C_\Lambda$  in place of  $C_\Lambda^{(k)}$ . We call  $k$  the resolution associated to  $L$ . An element  $[\Lambda]$  of  $C_\Lambda$  is written  $[\Lambda] = ([B], [S])$ . A  $\Lambda$ -bconfset (basic configuration set) is a subset of  $C_\Lambda$  of the form

$$C_{\Lambda, [B], [S]} := \{([B], [S]), [S] \in C_S\}, \quad (3.10)$$

or, stressing we work at resolution  $k$  (intervals of length  $e^{-L_k}$ ),

$$C_{\Lambda, [B]_{L_k}, [S]}^{(k)} := \left\{ ([B]_{L_k}, [S]_{L_k}), [S]_{L_k} \in C_S^{(k)} \right\}, \quad (3.11)$$

where  $[B] \in C_B$ .  $C_{\Lambda, [B], [S]}$  is a  $\Lambda$ -dense bconfset if the set of indices  $S$  satisfies the density condition

$$\#(S \cap \widehat{\Lambda}_{L^{1-}}) \geq L^{d-} \quad \text{for all boxes } \Lambda_{L^{1-}} \subset \Lambda_L. \quad (3.12)$$

Note that if  $S = \emptyset$  then  $B = \Lambda$ . We then set

$$C_{\Lambda,[B]} := C_{\Lambda,[B],\emptyset} = [B]. \quad (3.13)$$

We turn to random events. A  $\Lambda$ -bevent (basic event) is a subset of  $\Omega$  of the form

$$\mathcal{C}_{\Lambda,[B],S} := \{\omega \in \Omega, \{\omega\}_{i \in \Lambda} \in C_{\Lambda,[B],S}\}, \quad (3.14)$$

where  $[B] \in C_B$ , with the analog of (3.11) at resolution  $k$ .  $\mathcal{C}_{\Lambda,[B],S}$  is a  $\Lambda$ -dense bevent if  $S$  satisfies the density condition (3.12). In addition, we set

$$C_{\Lambda,[B]} := C_{\Lambda,[B],\emptyset} = \{\omega \in \Omega, \{\omega\}_{i \in \Lambda} \in C_{\Lambda,[B]}\} = \{\omega \in \Omega, \{\omega\}_{i \in \Lambda} = [B]\}. \quad (3.15)$$

Note that for each  $S_1 \subset S$  we have

$$C_{\Lambda,[B],S} = \bigsqcup_{[S_1] \in C_{S_1}} C_{\Lambda,[B] \sqcup [S_1], S \setminus S_1}, \quad (3.16)$$

$$C_{\Lambda,[B],S} = \bigsqcup_{[S_1] \in C_{S_1}} C_{\Lambda,[B] \sqcup [S_1], S \setminus S_1}, \quad (3.17)$$

where  $\sqcup$  denotes a disjoint union.

When changing scales, one redraws cylinders in the most natural way:

$$C_{\Lambda_{L_k}, [B]_{L_k}, S} = \bigsqcup_{[B]_{L_{k+1}} \subset [B]_{L_k}} C_{\Lambda_{L_k}, [B']_{L_{k+1}}, S}, \quad (3.18)$$

where  $[B]_{L_{k+1}} = \{I_{j_i}^{(k+1)}\}_i \subset [B]_{L_k} = \{I_{j_i}^{(k)}\}_i$  if and only if  $I_{j_i}^{(k+1)} \subset I_{j_i}^{(k)}$  for all  $i \in \Lambda_{k+1}$ .

**Definition 3.7.** Consider an energy  $E \in \mathbb{R}$ ,  $m > 0$ , and a box  $\Lambda = \Lambda_L(x)$ . The  $\Lambda$ -bevent  $\mathcal{C}_{\Lambda,[B],S}$  and the  $\Lambda$ -bconfset  $C_{\Lambda,[B],S}$  are  $(\Lambda, E, m)$ -good if the box  $\Lambda$  is  $([B], S, E, m)$ -good.

**Definition 3.8.** Consider an energy  $E \in \mathbb{R}$ , a rate of decay  $m > 0$ , and a box  $\Lambda$ . We call  $\Omega_\Lambda$  a  $(\Lambda, E, m)$ -localized event if there exist disjoint  $(\Lambda, E, m)$ -good dense bevents  $\{\mathcal{C}_{\Lambda,[B_i],S_i}\}_{i=1,2,\dots,I}$  such that

$$\Omega_\Lambda = \bigsqcup_{i=1}^I \mathcal{C}_{\Lambda,[B_i],S_i}. \quad (3.19)$$

We prove the following multiscale analysis.

**Proposition 3.9.** Let  $\varepsilon > 0$ ,  $\beta = 4/3+$  and  $p < \frac{3}{8}$  be given. There exists  $E_0 > 0$  and  $m > 0$  such that if  $L_1$  is large enough, then for all  $E \in [0, E_0]$  and all  $L \geq L_1$ , there exists an  $(\Lambda_L, E, m)$ -localized event  $\Omega_{\Lambda_L}$  with  $\mathbb{P}(\Omega_{\Lambda_L}) \geq 1 - L^{-pd}$ .

To prove Proposition 3.9, we use the Bourgain-Kenig multiscale analysis adapted to bevents as done in [GHK2]. The Wegner estimate of Bourgain-Kenig as stated in [BoK, Lemma 5.1 (and 5.1')] is translated into “bevents” language in [GHK2, Lemma 5.10].

Theorem 1.1 follows from Proposition 3.9 as in [GHK2, Section 6]. The initial condition can be obtained by the argument in [BoK, GHK2] using a large deviation result for  $\mu$  to estimate the probability that the averaged sum of the random variables in a cube of size, say,  $(\log L_0)^2$  is less than half its mean, where  $L_0$  is the initial length scale. We refer to [GK2] for a proof of the initial condition with an arbitrary measure  $\mu$ , as well as for dynamical localization.

It thus remains to prove [GHK2, Lemma 5.10] for our particular probability distribution  $\mu$ . This is the purpose of Section 5.

#### 4. MAXIMAL ANTICHAIN IN POSETS

In this section we briefly collect some tools and facts coming from the theory of posets (partially ordered multisets).

Let  $\mathcal{M} = \{1, 2, \dots, K\}^n$  be a multiset with partial order  $x \leq y$  iff  $x_i \leq y_i$  for all  $1 \leq i \leq n$ . Two elements  $x, y \in \mathcal{A}$  are comparable if  $x \leq y$  or  $y \leq x$ . We define the rank function  $r(x) = \sum x_i$  and the rank number  $N_r$  as the number of  $x \in \mathcal{M}$  with rank  $r$ . An antichain  $\mathcal{A} \subset \mathcal{M}$  is a collection of  $x$ 's such that no elements  $x, y \in \mathcal{A}$  are comparable. We further equip  $\mathcal{M}$  with the discrete uniform probability structure:  $\mathbb{P}_{\mathcal{M}}(x) = K^{-n}$ .

**Lemma 4.1.** *Let  $\mathcal{A} \subset \mathcal{M}$  an antichain, then  $\mathbb{P}_{\mathcal{M}}(\mathcal{A}) \leq c(K\sqrt{n})^{-1}$ .*

**Remark 4.2.** *Note that in our particular setting we will have  $K = 2^{k+1} \approx (\log n)^{\log 2 / \log \beta}$ , so the factor  $K^{-1}$  in (4.3) does not really improve the probability.*

*Proof.* We first recall the LYM inequality [A, Theorem 2.3.1]: It asserts that if  $\mathcal{A}$  is an antichain then

$$\sum_{x \in \mathcal{A}} \frac{1}{N_{r(x)}} \leq 1. \quad (4.1)$$

We combine (4.1) with an estimate on the maximal rank number [A, Theorem 4.3.6], namely, for some positive constants  $c, C$ ,

$$\frac{cK^n}{\sqrt{(K-1)(K+1)\sqrt{n}}} \leq \max_r N_r \leq \frac{CK^n}{\sqrt{(K-1)(K+1)\sqrt{n}}}. \quad (4.2)$$

We thus have

$$\mathbb{P}_{\mathcal{M}}(\mathcal{A}) = \frac{1}{K^n} \sum_{x \in \mathcal{A}} 1 \leq \frac{\max_r N_r}{K^n} \sum_{x \in \mathcal{A}} \frac{1}{N_{r(x)}} \leq \frac{\max_r N_r}{K^n} \lesssim \frac{1}{K\sqrt{n}}. \quad (4.3)$$

□

#### 5. THE WEGNER ESTIMATE

To prove [GHK2, Lemma 5.10] in our particular setting we need to prove the following lemma.

**Lemma 5.1.** *Consider a box  $\Lambda = \Lambda_L$ , let  $\ell = L^\rho$  with  $\rho = \beta^{-1} = \frac{3}{4}$  (so  $L^{\frac{4}{3}\rho} = L^{1-}$ ). Let  $\mathcal{C}_{\Lambda, [B], S} \subset \Omega$  be given. Let  $S \subset \widehat{\Lambda}$  with  $|S| = \ell^d$ , pick (and fix)  $\omega \in \mathcal{C}_{\Lambda, [B], S}$ , and set*

$$H(t_S) := H_{\omega, t_S, \Lambda} \quad \text{for all } t_S \in [0, 1]^S. \quad (5.1)$$

*Consider an energy  $E_0$ , set  $I = (E_0 - e^{-c_1\ell}, E_0 + e^{-c_1\ell})$ . Let  $E_\tau(\omega, t_S)$  be a continuous eigenvalue parametrization of  $\sigma(H(t_S))$  such that  $E_\tau(0) \in I$  (a finite family). Let  $E_\omega(t_S) = E_{\tau_0}(t_S)$  for some  $\tau_0$ , and  $E(\omega) = E_\omega(\omega_S)$ ,  $\omega_S = \{\omega\}_{i \in S}$ . Suppose*

$$e^{-c_3\ell^{\frac{4}{3}} \log \ell} \leq \frac{\partial}{\partial t_i} E(t_S) \leq e^{-c_2\ell} \quad \text{for all } i \in S \text{ if } E(t_S) \in I. \quad (5.2)$$

Then, there exists  $\Omega_{[B]} = \bigsqcup_l ([B], [S]_l) \subset \mathcal{C}_{\Lambda, [B], S}$  (the disjoint union being finite), such that

$$\left\{ \omega \in \mathcal{C}_{\Lambda, [B], S}; E(\omega) \in (E_0 - e^{-2c_3 \ell^{\frac{4}{3}} \log \ell}, E_0 + e^{-2c_3 \ell^{\frac{4}{3}} \log \ell}) \right\} \subset \mathcal{C}_{\Lambda, [B], S} \setminus \Omega_{[B]}, \quad (5.3)$$

and

$$\mathbb{P}(\mathcal{C}_{\Lambda, [B], S} \setminus \Omega_{[B]}; \mathcal{C}_{\Lambda, [B], S}) \leq C \ell^{-\frac{d}{2}+}. \quad (5.4)$$

*Proof.* Recall (3.9), that is we are given  $\varepsilon' > 0$  such that  $(1 + \varepsilon')(1 - \varepsilon) = 1$ . We have  $L \in [L_k^{1+\varepsilon'}, L_{k+1}^{1+\varepsilon'}]$  and thus

$$L_k \leq L^{1-\varepsilon} \leq L_{k+1}. \quad (5.5)$$

We set  $\ell = L^\rho$  and assume that  $\Lambda_\ell$  is a good box with probability  $\geq 1 - \ell^{-p}$ , as well as at previous scales. Let  $\omega \in \mathcal{C}_{\Lambda, [B], S}$  and assume that

$$E(\omega) \in (E_0 - e^{-2c_3 \ell^{\frac{4}{3}} \log \ell}, E_0 + e^{-2c_3 \ell^{\frac{4}{3}} \log \ell}). \quad (5.6)$$

We show that for any  $\omega'$  such that  $[\omega] = ([B], [S])$  and  $[\omega'] = ([B], [S]')$  are comparable and distinct, one has

$$E(\omega') \notin (E_0 - e^{-2c_3 \ell^{\frac{4}{3}} \log \ell}, E_0 + e^{-2c_3 \ell^{\frac{4}{3}} \log \ell}). \quad (5.7)$$

Indeed, suppose, for instance, that  $[\omega] < [\omega']$ . If  $[\omega_i] = [\omega'_i]$  then  $|\omega'_i - \omega_i| \leq e^{-L_{k+1}}$ , due to Definition 3.3. And if  $[\omega_i] < [\omega'_i]$  then by construction these two points are separated by a gap of length  $\geq \frac{1}{2}e^{-L_k}$  (for  $L$  large enough). Thanks to (5.2), we get

$$|E(\omega') - E(\omega)| \geq \frac{1}{2} e^{-c_3 \ell^{\frac{4}{3}} \log \ell} e^{-L_k} - \ell^{d-} e^{-c_2 \ell} e^{-L_{k+1}} \quad (5.8)$$

$$\geq \frac{1}{2} e^{-c_3 \ell^{\frac{4}{3}} \log \ell} e^{-L_k} \left( 1 - 4e^{-L_{k+1} + L_k} e^{c_3 \ell^{\frac{4}{3}} \log \ell} \right) \quad (5.9)$$

$$\geq \frac{1}{2} e^{-c_3 \ell^{\frac{4}{3}} \log \ell} e^{-L_k} \left( 1 - 4e^{-\frac{1}{4}L_{k+1}} \right) \quad (5.10)$$

$$\geq e^{-c_3 \ell^{\frac{4}{3}} \log \ell} \quad (5.11)$$

for  $L$  large enough, where we used that

$$c_3 \ell^{\frac{4}{3}} \log \ell \leq \frac{1}{4} L^{1-\varepsilon}. \quad (5.12)$$

We thus obtain (5.7). As a consequence  $E(\omega') \in (E_0 - e^{-2c_3 \ell^{\frac{4}{3}} \log \ell}, E_0 + e^{-2c_3 \ell^{\frac{4}{3}} \log \ell})$  can only happen for a set of  $\omega'$  such that the associated  $[S]$  and  $[S]'$  are non comparable, that is, for a collection of  $[S]$ 's that belongs to a finite antichain  $\mathcal{A}_{[B]} = \{[S]_1, [S]_2, \dots\}$ . Setting

$$\mathcal{C}_{\Lambda, [B], S} \setminus \Omega_{[B]} := ([B], \mathcal{A}_{[B]}) = \bigsqcup_l ([B], [S]_l), \quad (5.13)$$

we have proved (5.3). It remains to show the probabilistic estimate (5.4).

Let us give an element  $[S] = \left\{ I_{j_i}^{(k+1)} \right\}_{i \in S} \in \mathcal{C}_S$  and  $\omega \in \mathcal{C}_{\Lambda, [B], S}$ . By construction of  $\mu$ ,  $\mu(I_{j_i}^{(k+1)}) = K^{-1}$  for any  $i \in S$ , where  $K = 2^{k+1}$ . We thus replace the (continuous) probability space  $(\Omega, \mathbb{P})$  conditioned to  $\mathcal{C}_{\Lambda, [B], S}$  by the discrete

probability space  $(C_S, \mathbb{P}_S^o) \simeq (\{[S] \in C_S\}, \mathbb{P}_S^o)$ , where  $\mathbb{P}_S^o$  is the uniform measure with weight  $K^{-|S|}$ . We thus get

$$\mathbb{P} \{ \mathcal{C}_{\Lambda, [B], S} \setminus \Omega_{[B]}; \mathcal{C}_{\Lambda, [B], S} \} = \mathbb{P}_S^o(\mathcal{A}_{[B]}). \quad (5.14)$$

We now apply Lemma 4.1 to the set  $C_S \simeq \{1, 2, \dots, K\}^n$ , with  $K = 2^{k+1}$  and  $n = |S|$ , equipped with the discrete uniform probability  $\mathbb{P}_S^o$ . This ends the proof.  $\square$

## REFERENCES

- [AGKW] Aizenman, M., Germinet, F., Klein, A., Warzel, S.: On Bernoulli Decompositions for Random Variables, Concentration Bounds, and Spectral Localization. Submitted (2007)
- [A] Anderson, I.: *Combinatorics of finite sets*. Corrected reprint of the 1989 edition. Dover Publications Inc. Mineola, NY, 2002
- [Bo] Bourgain, J.: On localization for lattice Schrödinger operators involving Bernoulli variables. Geometric aspects of functional analysis. Lecture Notes in Math. **1850**, 77-99. Berlin: Springer, 2004
- [BoK] Bourgain, J., Kenig, C.: On localization in the continuous Anderson-Bernoulli model in higher dimension, Invent. Math. **161**, 389-426 (2005)
- [CoH1] Combes, J.M., Hislop, P.D.: Localization for some continuous, random Hamiltonians in d-dimension. J. Funct. Anal. **124**, 149-180 (1994)
- [CoH2] Combes, J.M., Hislop, P.D.: Landau Hamiltonians with random potentials: localization and the density of states. Commun. Math. Phys. **177**, 603-629 (1996)
- [CoHK] Combes, J.M., Hislop, P.D., Klopp, F.: Hölder continuity of the integrated density of states for some random operators at all energies. IMRN **4**, 179-209 (2003)
- [CoHK] Combes, J.M., Hislop, P.D., Klopp, F.: Optimal Wegner estimate. To appear in Duke Math. J.
- [CoHKN] Combes, J.M., Hislop, P.D., Klopp, F., Nakamura, S.: The Wegner estimate and the integrated density of states for some random operators. Spectral and inverse spectral theory (Goa, 2000), Proc. Indian Acad. Sci. Math. Sci. **112**, 31-53 (2002)
- [CoHKR] Combes, J.M., Hislop, P.D., Klopp, F., Raikov, G.: Global continuity of the integrated density of states for random Landau Hamiltonians. Commun. Part. Diff. Eqns. **29**, 1187-1214 (2004).
- [CoHM] Combes, J.M., Hislop, P.D., Mourre, E.: Spectral averaging, perturbation of singular spectra, and localization. Trans. Amer. Math. Soc. **348**, 4883-4894 (1996)
- [CoHN] Combes, J.M., Hislop, P.D., Nakamura, S.: The  $L^p$ -theory of the spectral shift function, the Wegner estimate and the integrated density of states for some random operators. Commun. Math. Phys. **218**, 113-130 (2001)
- [DrK] von Dreifus, H., Klein, A.: A new proof of localization in the Anderson tight binding model. Commun. Math. Phys. **124**, 285-299 (1989)
- [FK] Figotin, A., Klein, A.: Localization of classical waves I: Acoustic waves. Commun. Math. Phys. **180**, 439-482 (1996)
- [FrS] Fröhlich, J., Spencer, T.: Absence of diffusion with Anderson tight binding model for large disorder or low energy. Commun. Math. Phys. **88**, 151-184 (1983)
- [GHK1] Germinet, F., Hislop, P., Klein, A.: On localization for the Schrödinger operator with a Poisson random potential. C.R. Acad. Sci. Paris Ser. I **341**, 525-528 (2005)
- [GHK2] Germinet, F., Hislop, P., Klein, A.: Localization for the Schrödinger operator with a Poisson random potential. J. Europ. Math. Soc. **9**, 577-607 (2007)
- [GK1] Germinet, F., Klein, A.: Bootstrap Multiscale Analysis and Localization in Random Media. Commun. Math. Phys. **222**, 415-448 (2001)
- [GK2] Germinet, F., Klein, A.: Work in progress.
- [GKS] Germinet, F., Klein, A., Schenker, J.: Dynamical delocalization in random Landau Hamiltonians. To appear in Annals of Math.

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