ON THE QUANTIZATION OF HALL CURRENTS IN PRESENCE OF DISORDER

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Abstract. We review recent results of two of the authors concerning the quantization of Hall currents, in particular a general quantization formula for the difference of edge Hall conductances in semi-infinite samples with and without a confining wall. We then study the case where the Fermi energy is located in a region of localized states and discuss new regularizations. We also sketch the proof of localization for 2D-models with constant magnetic field with random potential located in a half-plane in two different situations: 1) with a zero potential in the other half plane and for energies away from the Landau levels and 2) with a confining potential in the other half plane and on an interval of energies that covers an arbitrary number of Landau levels.

1. The edge conductance and general invariance principles

Quickly after the discovery of the integer quantum Hall effect (IQHE) by von Klitzing et. al. [vK1], then Halperin [Ha] put the accent on the crucial role of quantum currents flowing at the edges of the (finite) sample. Such edge currents, carried by edge states, should be quantized, and the quantization should agree with the one of the transverse (Hall) conductance. While edge currents have been widely studied in the physics literature since the early eighties, e.g. [MDS, AS, FGK, ZMH, CFGP] (see also [PG, vK2] and references therein), it is only recently that a mathematical understanding of the existence of such edge currents has been obtained [DBP, FGW, EJK1, EJK2, CHS1, FM, CHS2]. The study of the quantization of the edge Hall conductance at a mathematical level is even more recent [SBKR, KRSB, KSB, EG, CG, EGS].

We consider the simplest model for quantum devices exhibiting the IQHE. This consists of an electron confined to the 2-dimensional plane considered as the union of two complementary semi-infinite regions supporting potentials V_1 and V_2, respectively, and under the influence of a constant magnetic field B orthogonal to the sample. In the absence of potentials V_1 and V_2, the free electron is described by the free Landau Hamiltonian $H_L = p_x^2 + (p_y - Bx)^2$. The spectrum of H_L consists of the well-known Landau levels $B_N = (2N-1)B$, $N \geq 1$, with the convention $B_0 = -\infty$. To introduce the half-plane potentials, we let $\mathbf{1}_-$ and $\mathbf{1}_+$ be the characteristic functions of, respectively, $\{x \leq 0\}$ and $\{x > 0\}$. Then, if V_1, V_2 are two potentials bounded from below and in the Kato class [CFKS], the Hamiltonian of the system is given, in suitable units and Landau gauge, by

$$H(V_1, V_2) := H_L + V_1 \mathbf{1}_- + V_2 \mathbf{1}_+ \,, \tag{1.1}$$

as a self-adjoint operator acting on $\mathrm{L}^2(\mathbb{R}^2, \mathrm{d}x\mathrm{d}y)$, where $H_L = H(0,0)$ in this notation. For technical reasons it is convenient to assume that V_1, respectively V_2, does not grow faster than polynomially as $x \to -\infty$, respectively, as $x \to +\infty$.

We shall say that V_1 is a left *confining potential* with respect to the interval $I = [a, b] \subset \mathbb{R}$ if, in addition to the previous conditions, the following holds: There exists $R > 0$, s.t.

$$\forall x \leq -R,\ \forall y \in \mathbb{R},\ V_1(x, y) > b. \tag{1.2}$$

The "hard wall" case, i.e. $V_1 = +\infty$ and $H = H_L + V_2$ acting on $\mathrm{L}^2(\mathbb{R}^+ \times \mathbb{R}, \mathrm{d}x\mathrm{d}y)$ with Dirichlet boundary conditions, can be considered as well.

As typical examples for $H(V_1, V_2)$ one may think of the right potential V_2 as an impurity potential and the left potential V_1 as either a wall, confining the electron to the right half-plane and generating an edge current near $x = 0$, or as the zero potential. In this latter case, the issue is to determine whether or not V_2 is strong enough to create edge currents by itself. We will discuss this in Section 3.1. Another example is the strip geometry, where both V_1 and V_2 are confining potentials outside of a strip $x \in [-R, R]$, where the electron is localized.

We define a "switch" function as a smooth real valued *increasing* function equal to 1 (resp. 0) at the right (resp. left) of some bounded interval. Following [SBKR, KRSB, EG, CG], we define the (Hall) edge conductance as follows.

Definition 1.1. *Let $\mathcal{X} \in \mathcal{C}^\infty(\mathbb{R}^2)$ be a x-translation invariant switch function with* $\mathrm{supp}\mathcal{X}' \subset \mathbb{R} \times [-\frac{1}{4}, \frac{1}{4}]$, *and let $-g \in \mathcal{C}^\infty(\mathbb{R})$ be switch a function with* $\mathrm{supp}\, g' \subset I = [a, b]$ *a compact interval. The* edge conductance *of $H(V_1, V_2)$ in the interval I is defined as*

$$\sigma_e(g, V_1, V_2) = -\mathrm{tr}(g'(H(V_1, V_2))i[H_L, \mathcal{X}]) \tag{1.3}$$

whenever the trace is finite (we shall also use the notation $\sigma_e(g, H) = \sigma_e(g, V_1, V_2)$ if $H = H(V_1, V_2)$).

Note that in the situations of interest $\sigma_e(g, V_1, V_2)$ will turn out to be independent of the particular shape of the switch function $\mathcal{X}$ and also of the switch function g, provided supp g' does not contain any Landau level.

We turn to the description of the results of [CG].

Let us assume that I lies in between two successive Landau levels, say the N^{th} and the $(N{+}1)^{\mathrm{th}}$. While clearly $\sigma_e(g, 0, 0) = 0$, for any g as above since $g'(H_L) = 0$, a straightforward computation shows that $\sigma_e(g, V_1, 0) = N$, provided $V_1(x_1, x_2) = V_1(x_1)$ is such that $\lim_{x_1 \to -\infty} V_1(x_1) > b$ (see, for example, [CG, Proposition 1]). The first result tells us that the edge conductance is stable under a perturbation by a potential W located in a strip $[L_1, L_2] \times \mathbb{R}$ of finite width.

Theorem 1.2. ([CG, Theorem 1]) *Let $H = H(V_1, V_2)$ be as in* (1.1)*, and let W be a bounded potential supported in a strip $[L_1, L_2] \times \mathbb{R}$, with $-\infty < L_1 < L_2 < +\infty$. Then the operator $(g'(H + W) - g'(H))i[H_L, \mathcal{X}]$ is trace class, and*

$$\mathrm{tr}((g'(H + W) - g'(H))i[H_L, \mathcal{X}]) = 0. \tag{1.4}$$

As a consequence:
(i) $\sigma_e(g, H_L + W) = 0$.
(ii) Assume V_1 is a y-invariant potential, i.e. $V_1(x, y) = V_1(x)$, that is left confining with respect to $I \supset$ supp g'. *If $I \subset]B_N, B_{N+1}[$, for some $N \geq 0$, then*

$$\sigma_e(g, H_L + V_1 + W) = N. \tag{1.5}$$

We note that Theorem 1.2 extends perturbations W that decay polynomially fast in the x-direction. In particular, it allows for more general confining potentials than y-invariant ones. But, it is easy to see that Theorem 1.2 does not hold for all perturbations in the x direction. For example, if supp $g' \subset]B_N, B_{N+1}[$, then $\sigma_e(g,0,0) = 0$, so for if $W_\ell(x,y) = \nu_0 \mathbf{1}_{[0,\ell]}$, with $\nu_0 > B_{N+1}$ and $0 < \ell < \infty$, then $\sigma_e(g,0,W_\ell) = 0$. On the other hand, a simple calculation shows (e.g. [CG]) that for $W_\infty = \nu_0 \mathbf{1}_{[0,\infty[}$, we have $\sigma_e(g,0,W_\infty) = -N$. However, one has the following invariance principle, which is a consequence of a more general sum rule given in [CG, Theorem 2].

Theorem 1.3. ([CG, Corollary 3] *Let g be s.t.* supp $g' \subset]B_N, B_{N+1}[$, *for some $N \geq 0$. Let V_1 be a y-invariant left confining potential with respect to* suppg'. *Then the operator $\{g'(H(V_1,V_2)) - g'(H(0,V_2))\} i[H_L, \mathcal{X}]$ is trace class and*

$$-\mathrm{tr}(\{g'(H(V_1,V_2)) - g'(H(0,V_2))\} i[H_L, \mathcal{X}]) = N. \tag{1.6}$$

In particular, if either $\sigma_e(g,V_1,V_2)$ or $\sigma_e(g,0,V_2)$ is finite, then both are finite, and

$$\sigma_e(g,V_1,V_2) - \sigma_e(g,0,V_2) = N. \tag{1.7}$$

An immediate, but important, consequence of Theorem 1.3, if $\|V_2\|_\infty < B$, then $\sigma_e(g,V_1,V_2) = N$, whenever $I \subset]B_N + \|V_2\|_\infty, B_{N+1} - \|V_2\|_\infty[$, recovering a recent result of Kellendonk and Schulz-Baldes [KSB].

In general, neither term in (1.6) is separately trace class but a meaning can be given to each term through an appropriate regularization. Various regularizations of the edge conductance were discussed in [CG], and two others are presented in Section 2. It is proved in [CG] that the regularized edge conductance $\sigma_e^{reg}(g,V_0,V)$ satisfies a sum rule similar to (1.7):

$$\sigma_e^{\mathrm{reg}}(g,V_0,V) = N + \sigma_e^{\mathrm{reg}}(g,0,V). \tag{1.8}$$

With reference to this, note that $\sigma_e^{reg}(g,0,V) \neq 0$ would imply the existence of current carrying states solely due to the impurity potential. Since (1.8) would yield $\sigma_e(g,V_1,V_2) \neq N$, we see that such "edge currents without edges" are responsible for the deviation of the Hall conductance from its ideal value N. Typically this is expected to happen in a regime of strong disorder (with respect to the magnetic strength B). As an example of this phenomenon, a model studied by S. Nakamura and J. Bellissard [NB] is reviseted in [CG] and it is shown that in this case $\sigma_e(g,V_0,V_2) = 0$ and thus $\sigma_e(g,0,V_2) = -N$. In Section 3.2, we present another example for which localization in the strong disorder regime implies that $\sigma_e^{reg}(0,V)$ is quantized so that there are edge currents without edges. As a counterpart, in the weak disorder regime (i.e. weak impurities in the region $x_1 \geq 0$ and no electric potential in the left half-plane), one expects that no current will flow near the region $x_1 = 0$. After a regularizing procedure, we argue in Section 3.1 that this is exactly what happens with the model studied in [CH, GK2, Wa] if I lies in a region of the spectrum where localization has been shown.

Notation: Throughout this note $\mathbf{1}_X = \mathbf{1}_{(x,y)}$ will denote the characteristic function of a unit cube centered $X = (x,y) \in \mathbb{Z}^2$. If A is a subset of $\mathbb{R}^2$, then $\mathbf{1}_A$ will denote the characteristic function of this set. We recall that $\mathbf{1}_-$ and $\mathbf{1}_+$ stand, respectively, for $\mathbf{1}_{x\leq 0}$ and $\mathbf{1}_{x>0}$.

2. Regularizing the edge conductance in presence of impurities

2.1. Generalities.

Let $V_2 = V$ be a potential located in the region $x \geq 0$. If the operator $H(0,V)$ has a gap and if the interval I falls into this gap, then the edge conductance is quantized as mentioned above. However such a situation is not physically relevant, since the quantization of the Hall conductance can only be related to the quantum Hall effect in presence of impurities that create the well-known “plateaus” [BESB, vK2]. If I falls into a region of localized states of $H(0,V)$, then the conductances may not be well-defined, and a regularization is needed. In this section, we briefly recall the regularization procedure described in [CG], and we then propose new candidates.

Assume supp $g' \subset I \subset]B_N, B_{N+1}[$. Let $(J_R)_{R>0}$ be a family of operators s.t.

C1. $\|J_R\| = 1$ and $\lim_{R\to\infty} J_R\psi = \psi$, for all $\psi \in E_{H(0,V)}(I)\mathrm{L}^2(\mathbb{R}^2)$.

C2. J_R regularizes $H(0,V)$ in the sense that $g'(H(0,V))i[H_L,\mathcal{X}]J_R$ is trace class for all $R > 0$, and $\lim_{R\to\infty} \mathrm{tr}(g'(H(0,V))i[H_L,\mathcal{X}]J_R)$ exists and is finite.

Then if $V_1 = V_0$ is a y-invariant left confining potential with respect to I, it follows from Theorem 1.3 that

$$\lim_{R\to\infty} -\mathrm{tr}\left(\{g'(H(V_0,V)) - g'(H(0,V))\} i[H_L,\mathcal{X}]J_R\right) = N.$$

In other terms, if **C1** and **C2** hold, then J_R also regularizes $H(V_0,V)$. Defining the regularized edge conductance by

$$\sigma_e^{\mathrm{reg}}(g,V_1,V_2) := -\lim_{R\to\infty} \mathrm{tr}(g'(H(V_1,V_2))i[H_L,\mathcal{X}]J_R), \tag{2.1}$$

whenever the limit exists, we get the analog of Theorem 1.3:

$$\sigma_e^{\mathrm{reg}}(g,V_0,V) = N + \sigma_e^{\mathrm{reg}}(g,0,V). \tag{2.2}$$

In particular, if we can show that $\sigma_e^{\mathrm{reg}}(g,0,V) = 0$, for instance, under some localization property, then the edge quantization for $H(V_0,V)$ follows:

$$\sigma_e^{\mathrm{reg}}(g,V_0,V) = -\lim_{R\to\infty} \mathrm{tr}(g'(H(V_0,V))i[H_L,\mathcal{X}]J_R) = N. \tag{2.3}$$

Let us now consider

$$H_\omega = H(0,V_{\omega,+}) = H_L + V_{\omega,+}, \quad V_{\omega,+} = \sum_{i\in\mathbb{Z}^{+*}\times\mathbb{Z}} \omega_i u(x-i), \tag{2.4}$$

a random Schrödinger operator modeling impurities located on the positive half-plane (the $(\omega_i)_i$ are i.i.d. (independent, identically distributed) random variables, and u is a smooth bump function). If H_ω has pure point spectrum in I for $\mathbb{P}$-a.e. ω, then denoting by $(\varphi_{\omega,n})_{n\geq 1}$ a basis of orthonormalized eigenfunctions of H_ω with energies $E_{\omega,n} \in$ supp $g' \subset I$, one has, whenever the regularization holds,

$$\sigma_e^{\mathrm{reg}}(g,0,V_{\omega,+}) = -\lim_{R\to\infty} \sum_n g'(E_{\omega,n})\langle\varphi_{\omega,n}, i[H_\omega,\mathcal{X}]J_R\varphi_{\omega,n}\rangle. \tag{2.5}$$

If $J_R = \mathbf{1}_{x\leq R}$, the limit (2.5) actually exists [CG, Propostion 2], but it is very likely that it will not be zero, even under strong localization properties of the eigenfunctions such as (SULE) (see [DRJLS]) or (SUDEC) (see Definition 2.3 below and [GK3]). We refer to [EGS] for a concrete example. This can be understood as follows: Because the cut-off J_R (even a smooth version of it) cuts classical orbits living near $x = R$, it will create spurious contributions to the total current, and the latter will no longer be zero. The quantum counter part of this picture is that although the expectation of $i[H_\omega,\mathcal{X}]$ in an eigenstate of H_ω is zero by the Virial Theorem, this is no longer true if this commutator is multiplied by J_R. Of course, the sum in (2.5) is zero if J_R commutes with H_ω, as in [CG, Theorem 3]. In the next two sections we investigate new regularizations that commute with H_ω only asymptotically (as $R \to \infty$).

2.2. **A time averaged regularization for a dynamically localized system.** We assume that the operator $H = H(0, V)$ exhibits dynamical localization in an open interval $I \in]B_N, B_{N+1}[$. This means that for any $p \geq 0$, there exists a nonnegative constant $C_p < \infty$ such that for any Borel function f on I, with $|f| \leq 1$, and for any $X_1, X_2 \in \mathbb{R}^2$,

$$\sup_{t\in\mathbb{R}} \|\mathbf{1}_{X_1} f(H) e^{-itH} \mathbf{1}_{X_2}\|_2 \leq C_p \min(1, |X_1 - X_2|)^{-p}). \tag{2.6}$$

We used the Hilbert-Schmidt norm. For random Schrödinger operators H_ω, this assumption is one of the standard conclusions of multiscale analysis [GDB, GK1]. We show in Section 3.1 that as long as $I \in]B_N, B_{N+1}[$ such an analysis applies to the Hamiltonian $H_\omega = H(0, V_\omega)$ as in (2.4).

Note that it follows from (2.6) that if $X \in \mathbb{R}^2$ and A is a subset of $\mathbb{R}^2$ (A may contain X) then, for any $p > 0$, there exists a (new) constant $0 \leq C_p < \infty$, such that

$$\sup_{t\in\mathbb{R}} \|\mathbf{1}_A f(H) e^{-itH} \mathbf{1}_X\|_2 \leq C_p \min(1, \mathrm{dist}(\{X\}, A)^{-p}). \tag{2.7}$$

For $R < \infty$, $\eta > 0$ and $\gamma > 0$, we set, with $H = H(0, V)$ and $X = (x, y)$,

$$J_R = \eta \int_0^\infty E_H(I) e^{itH} \mathbf{1}_{x\leq R} e^{-itH} E_H(I) e^{-\eta t} dt, \text{ with } R = \eta^{-\gamma}. \tag{2.8}$$

Theorem 2.1. *Let J_R as in* (2.8)*, with $\gamma \in]0, 1[$. Assume that $H(0, V)$ exhibits dynamical localization (i.e.* (2.6)*) in $I \subset]B_N, B_{N+1}[$ for some $N \geq 0$. Then J_R regularizes $H(0, V)$, and thus also $H(V_0, V)$, in the sense that **C1** and **C2** hold. Moreover the edge conductances take the quantized values: $\sigma_e^{\mathrm{reg}}(g, 0, V) = 0$ and $\sigma_e^{\mathrm{reg}}(g, V_0, V) = N$.*

Remark 2.2. *In* [EGS]*, a similar regularization is considered, where $\gamma = 1$ and H is the* bulk *Hamiltonian $H(V, V)$. We also note that if R and η are independent variables, then one recovers the regularization* [CG, Eq. (7.13)]*, see* [CG, Remark 13].

Proof. Since $\|J_R\| \leq 1$, to get **C1** it is enough to check $\lim_{R\to\infty} J_R E_H(I)\psi E_H(I)\psi$ for compactly supported states, and it is thus enough to note that by (2.7),

$$\|(1 - J_R) E_H(I) \mathbf{1}_0\| \leq \eta \int_0^\infty \sup_t \|\mathbf{1}_{x>R} e^{-itH} E_H(I) \mathbf{1}_0\| e^{-\eta t} dt \leq C_p R^{-p}. \tag{2.9}$$

We turn to **C2**. As in [CG], we write $i[H, \mathcal{X}] = i[H, \mathcal{X}]\mathbf{1}_{|y|\leq\frac{1}{2}}$, with $\mathbf{1}_{|y|\leq\frac{1}{2}} = \sum_{x_2\in\mathbb{Z}} \mathbf{1}_{(x_2,0)}$. Note that, by hypothesis on I, $g'(H) = g'(H) - g'(H_L)$, so that terms that are far in the left half plane will give small contributions. To see this, we develop

$$\left\|(g'(H) - g'(H_L)) i[H_L, \mathcal{X}] E_H(I) e^{-itH} \mathbf{1}_{x\leq -R}\right\|_1 \tag{2.10}$$

using the Helffer-Söjstrand formula [HeSj, HuSi] and the resolvent identity with $R_L(z) = (H_L - z)^{-1}$ and $R(z) = (H - z)^{-1}$. It is thus enough to control terms of the form, with $\mathrm{Im} z \neq 0$, $x_1, y_1, x_2 \in \mathbb{Z}$, $X_1 = (x_1, y_1)$,

$$\left\|R(z) V \mathbf{1}_{x\geq 0} R_L(z) i[H_L, \mathcal{X}] \mathbf{1}_{|y|\leq\frac{1}{2}} E_H(I) e^{-itH} \mathbf{1}_{x\leq -R}\right\|_1 \tag{2.11}$$

$$\leq \sum_{x_1\geq 0, y_1, x_2} \|R(z) V \mathbf{1}_{X_1}\|_2 \|\mathbf{1}_{X_1} R_L(z) i[H_L, \mathcal{X}] \mathbf{1}_{(x_2,0)}\| \|\mathbf{1}_{(x_2,0)} E_H(I) e^{-itH} \mathbf{1}_{x\leq -R}\|_2$$

But, as a well-known fact, $R(z)\mathbf{1}_{X_1}$ is Hilbert-Schmidt in dimension 2, e.g. [GK2, Lemma A.4], and its Hilbert-Schmidt norm is bounded by $C[\mathrm{dist}(z,\sigma(H))]^{-1} \leq C\Im z^{-1}$, uniformly in X_1. Then, for some $\kappa \geq 1$,

$$\begin{aligned}
(2.11) \quad &\leq \frac{C}{(\Im z)^\kappa} \sum_{x_1\geq 0, y_1, x_2 \leq -\frac{R}{2}} \|\mathbf{1}_{X_1} R_L(z) i[H_L,\mathcal{X}]\mathbf{1}_{(x_2,0)}\| && (2.12)\\
&\quad + \frac{C}{(\Im z)^\kappa} \sum_{x_2 \geq -\frac{R}{2}} \|\mathbf{1}_{(x_2,0)} E_H(I) e^{-itH}\mathbf{1}_{x\leq -R}\|_2 && (2.13)\\
&\leq \frac{C}{(\Im z)^\kappa} R^{-p}, && (2.14)
\end{aligned}$$

where the latter follows from (2.7) and from the fact that $\|\mathbf{1}_{X_1} R_L(z) i[H_L,\mathcal{X}]\mathbf{1}_{(x_2,0)}\|$ decays faster than any polynomial in $|X_1 - X_2|$, as can be seen by a Combes-Thomas estimate together with standard computations (e.g. [CG, Lemma 3]). As a consequence the trace norm (2.10) is finite and goes to zero as $R \to \infty$, uniformly in η.

The next step is to control contributions coming from terms living far from the support of $\mathcal{X}'$, i.e. terms s.t. $|y| \geq R^\nu$ with $\nu \in]0,1]$. Set $S(R,\nu) = \{(x,y) \in \mathbb{R}^2,\ |x| \leq R, |y| \geq R^\nu\}$. Then, using (2.7),

$$\begin{aligned}
&\left\|g'(H) i[H_L,\mathcal{X}] E_H(I) e^{-itH}\mathbf{1}_{S(R,\nu)}\right\|_1 && (2.15)\\
&\leq \sum_{x_2\in\mathbb{Z}} \left\|g'(H) i[H_L,\mathcal{X}]\mathbf{1}_{(x_2,0)}\right\|_2 \left\|\mathbf{1}_{(x_2,0)} E_H(I) e^{-itH}\mathbf{1}_{S(R,\nu)}\right\|_2 && (2.16)\\
&\leq C\sum_{x_2\in\mathbb{Z}} \min(1, \mathrm{dist}(\{(x_2,0)\}, S(R,\nu))^{-p}) \leq C(2R)(R^\nu)^{-p}. && (2.17)
\end{aligned}$$

Contributions from (2.15) are thus negligible as $R \to \infty$, uniformly in η. Letting $K_{R,\nu}$ denote the compact set

$$K_{R,\nu} = \{(x,y) \in \mathbb{R}^2, |x| \leq R, |y| \leq R^\nu\},$$

we are left so far with the evaluation of

$$\eta \int_0^\infty e^{-\eta t} dt\, g'(H) i[H_L,\mathcal{X}] E_H(I) e^{itH}\mathbf{1}_{K_{R,\nu}} e^{-itH} E_H(I), \tag{2.18}$$

which is clearly now a trace class operator (the integral is absolutely convergent in trace norm). In other terms $g'(H)i[H_L,\mathcal{X}]J_R$ is thus trace class. It remains to show that its trace goes to zero as R goes to infinity. But on the account of (2.10) and (2.15), it remains to show that the trace of (2.18) goes to zero. By cyclicity,

$$\begin{aligned}
\mathrm{tr}(2.18) &= -\eta\int_0^\infty e^{-\eta t}\mathrm{tr}\left\{g'(H)\mathcal{X} e^{-itH} i[H_L, E_H(I)\mathbf{1}_{K_{R,\nu}} E_H(I)] e^{itH}\right\} \mathrm{d}t\\
&= \eta\int_0^\infty e^{-\eta t}\frac{d}{dt}\mathrm{tr}\left\{g'(H)\mathcal{X} e^{-itH} E_H(I)\mathbf{1}_{K_{R,\nu}} E_H(I) e^{itH}\right\} \mathrm{d}t\\
&= \eta\mathrm{tr}\left\{g'(H)\mathcal{X} E_H(I)\mathbf{1}_{K_{R,\nu}} E_H(I)\right\}\\
&\quad -\eta^2\int_0^\infty e^{-\eta t}\mathrm{tr}\left\{g'(H)\mathcal{X} e^{-itH} E_H(I)\mathbf{1}_{K_{R,\nu}} E_H(I) e^{itH} E_H(I)\right\} \mathrm{d}t.
\end{aligned}$$

Thus $|\mathrm{tr}(2.18)| \leq C\eta |K_{R,\nu}| = C\eta R^{1+\nu}$. Since $R = \eta^{-\gamma}$, the trace goes to zero if $\gamma < \frac{1}{1+\nu}$. □

2.3. **Regularization under a stronger form of dynamical localization.**

In this section, we consider

$$J_R = E_{H(0,V)}(I)\mathbf{1}_{x\le R}E_{H(0,V)}(I). \tag{2.19}$$

Note that the regularization (2.8) studied in Section (2.2) is the time average of (2.19). The effect of the time averaging is to provide a control on the cross terms arising in (2.19) if one expands $E_{H(0,V)}(I)$ over a basis of eigenfunctions. In [CG, Eq. (7.13)], cross terms were suppressed from the very definition of J_R. By showing that J_R, given in (2.19), regularizes $H(0,V_\omega)$ under, basically, the same assumption as in [CG, Theorem 3], we strengthen [CG]'s result.

Let T be the multiplication operator by $T(X) = \langle X\rangle^\nu$, $\nu > \frac{d}{2} = 1$, with $\langle X\rangle = (1+|X|^2)^{\frac{1}{2}}$, for $X \in \mathbb{R}^2$. It is well known for Schrödinger operators that $\mathrm{tr}(T^{-1}E_{H(0,V)}(I)T^{-1}) < \infty$, if I is compact (e.g. [GK2]).

Definition 2.3 (SUDEC)**.** *Assume H has pure point spectrum in I with eigenvalues E_n and corresponding normalized eigenfunctions φ_n, listed with multiplicities. We say that H has* Summable Uniform Decay of Eigenfunction Correlations *(SUDEC) in I, if there exist $\zeta \in]0,1[$ and a finite constant $c_0 > 0$ such that for any $E_n \in I$ and $X_1, X_2 \in \mathbb{Z}^2$,*

$$\|\mathbf{1}_{X_1}\varphi_n\|\|\mathbf{1}_{X_2}\varphi_n\| \le c_0\alpha_n\|T\mathbf{1}_{X_1}\|^2\|T\mathbf{1}_{X_2}\|^2\mathrm{e}^{-|X_1-X_2|^\zeta}, \tag{2.20}$$

where $\alpha_n = \|T^{-1}\varphi_n\|^2$.

Note that,

$$\sum_n \alpha_n = \mathrm{tr}(T^{-1}E_{H(0,V)}(I)T^{-1}) < \infty. \tag{2.21}$$

Remark 2.4. *Property* (2.20) *(or a modified version of it) was called (WULE) in* [CG] *and was introduced in* [Ge]*. The more accurate acronym (SUDEC) comes from* [GK3] *and Property (SUDEC) is used in* [GKS] *as a very natural signature of localization in order to get the quantization of the bulk conductance.*

Theorem 2.5. *Assume that $H(0,V)$ has (SUDEC) in $I \subset]B_N, B_{N+1}[$ for some $N \ge 0$. Then J_R, given in* (2.19)*, regularizes $H(0,V)$, and thus also $H(V_0,V)$, in the sense that **C1** and **C2** hold. Moreover the edge conductances take the quantized values: $\sigma_e^{\mathrm{reg}}(g,0,V) = 0$ and $\sigma_e^{\mathrm{reg}}(g,V_0,V) = N$.*

Proof. That the operator $g'(H(0,V))i[H(0,V),\mathcal{X}]J_R$ is trace class follows from the comparison $g'(H(0,V)) = g'(H(0,V)) - g'(H_L)$. In order to control the region $x \le 0$, and the immediate estimate, let P_n be the eigenprojector on the eigenfunction φ_n, and write

$$\begin{aligned}\|\mathbf{1}_X E_{H(0,V)}(I)\mathbf{1}_Y\|_2 &\le \sum_n \|\mathbf{1}_X P_n \mathbf{1}_Y\|_2 = \sum_n \|\mathbf{1}_X\varphi_n\|\|\mathbf{1}_Y\varphi_n\| &(2.22)\\ &\le c_0(\sum_n \alpha_n)\|T\mathbf{1}_X\|^2\|T\mathbf{1}_Y\|^2\mathrm{e}^{-|X-Y|^\zeta}, &(2.23)\end{aligned}$$

where we used the assumption (2.20) (and recall (2.21)). We proceed and set $\Lambda_{2,R} = \mathbf{1}_{x\le R}$. We are looking at

$$\sigma_E^{(reg)}(g,R) = \mathrm{tr}(g'(H(0,V))i[H(0,V),\mathcal{X}]J_R). \tag{2.24}$$

The operator being trace class, we expand the trace in the basis of eigenfunctions of $H(0,V)$ in the interval I and get

$$|\sigma_E^{(reg)}(g,R)| = |\sum_{n\neq m} g'(E_n)(E_n-E_m)\langle\varphi_n,\mathcal{X}\varphi_m\rangle\langle\varphi_m,\Lambda_{2,R}\varphi_n\rangle| \quad (2.25)$$

$$\leq C(g,I)\sum_{n\neq m}|\langle\varphi_n,\mathcal{X}\varphi_m\rangle||\langle\varphi_m,\Lambda_{2,R}\varphi_n\rangle|. \quad (2.26)$$

It remains to show that the double sum in (2.26) is convergent (i.e. the trace is absolutely convergent). If it is so, then we can interchange the limit in R and the double sum to get zero due to the orthogonality of the eigenfunctions. In full generality, dynamical localization is not enough to show that (2.26) is absolutely convergent. This is the case if $H = H(0,0) = H_L$ and I contains a Landau level. The sum (2.26) will not converge, even though H_L exhibits dynamical localization. But if one has (SULE) or (SUDEC), then the sum converges absolutely. We have, writing Λ_2 instead of $\Lambda_{2,R}$:

$$(2.26) \quad (2.27)$$

$$\leq \sum_{n\neq m}|\langle\varphi_n,\mathcal{X}\varphi_m\rangle|\,|\langle\varphi_m,\Lambda_2\varphi_n\rangle| \quad (2.28)$$

$$= \sum_{n\neq m}|\langle\varphi_n,\mathcal{X}\varphi_m\rangle|^{\frac12}|\langle\varphi_n,(1-\mathcal{X})\varphi_m\rangle|^{\frac12}|\langle\varphi_m,\Lambda_2\varphi_n\rangle|^{\frac12}|\langle\varphi_m,(1-\Lambda_2)\varphi_n\rangle|^{\frac12}$$

$$\leq \sum_{n\neq m}\Big(\|\sqrt{\mathcal{X}}\varphi_n\|\|\sqrt{\mathcal{X}}\varphi_m\|\|\sqrt{1-\mathcal{X}}\varphi_n\|\|\sqrt{1-\mathcal{X}}\varphi_m\| \quad (2.29)$$

$$\times\|\sqrt{\Lambda_2}\varphi_n\|\|\sqrt{\Lambda_2}\varphi_m\|\|\sqrt{1-\Lambda_2}\varphi_n\|\|\sqrt{1-\Lambda_2}\varphi_m\|\Big)^{\frac12} \quad (2.30)$$

$$\leq \sum_n\Big(\|\sqrt{\mathcal{X}}\varphi_n\|\|\sqrt{1-\mathcal{X}}\varphi_n\|\|\sqrt{\Lambda_2}\varphi_n\|\|\sqrt{1-\Lambda_2}\varphi_n\|\Big)^{\frac12} \quad (2.31)$$

$$\times\sum_m\Big(\|\sqrt{\mathcal{X}}\varphi_m\|\|\sqrt{1-\mathcal{X}}\varphi_m\|\|\sqrt{\Lambda_2}\varphi_m\|\|\sqrt{1-\Lambda_2}\varphi_m\|\Big)^{\frac12} \quad (2.32)$$

It remains to show that (2.20) implies

$$\sum_n\Big(\|\sqrt{\mathcal{X}}\varphi_n\|\|\sqrt{1-\mathcal{X}}\varphi_n\|\|\sqrt{\Lambda_2}\varphi_n\|\|\sqrt{1-\Lambda_2}\varphi_n\|\Big)^{\frac12}<\infty. \quad (2.33)$$

We consider division of $\mathbb{R}^2$ into four quadrants given by the supports of the various localization functions: I = supp $\mathcal{X}\Lambda_2$, II = supp $(1-\mathcal{X})\Lambda_2$, III = supp $(1-\mathcal{X})(1-\Lambda_2)$, and IV = supp $\mathcal{X}(1-\Lambda_2)$. We first note that summing (2.20) over two opposite quadrants (**I**)(**III**) yields a constant:

$$\|\sqrt{\mathcal{X}}\sqrt{\Lambda_2}\varphi_n\|\|\sqrt{1-\mathcal{X}}\sqrt{1-\Lambda_2}\varphi_n\|\leq c\alpha_n, \quad (2.34)$$

and summing over the opposite quadrants (**II**)(**IV**) yields,

$$\|\sqrt{\mathcal{X}}\sqrt{1-\Lambda_2}\varphi_n\|\|\sqrt{1-\mathcal{X}}\sqrt{\Lambda_2}\varphi_n\|\leq c\alpha_n \quad (2.35)$$

We write a term in (2.33) as

$$\begin{aligned}
&\|\sqrt{\mathcal{X}}\varphi_n\|\|\sqrt{1-\mathcal{X}}\varphi_n\|\|\sqrt{\Lambda_2}\varphi_n\|\|\sqrt{1-\Lambda_2}\varphi_n\| && (2.36)\\
\leq\ & (\|\sqrt{\mathcal{X}}\sqrt{\Lambda_2}\varphi_n\|+\|\sqrt{\mathcal{X}}\sqrt{1-\Lambda_2}\varphi_n\|)\\
& (\|\sqrt{1-\mathcal{X}}\sqrt{\Lambda_2}\varphi_n\|+\|\sqrt{1-\mathcal{X}}\sqrt{1-\Lambda_2}\varphi_n\|)\\
& (\|\sqrt{\mathcal{X}}\sqrt{\Lambda_2}\varphi_n\|+\|\sqrt{1-\mathcal{X}}\sqrt{\Lambda_2}\varphi_n\|)\\
& (\|\sqrt{\mathcal{X}}\sqrt{1-\Lambda_2}\varphi_n\|+\|\sqrt{1-\mathcal{X}}\sqrt{1-\Lambda_2}\varphi_n\|)\\
=\ & (\mathbf{I}+\mathbf{IV})(\mathbf{II}+\mathbf{III})(\mathbf{I}+\mathbf{II})(\mathbf{III}+\mathbf{IV}). && (2.37)
\end{aligned}$$

This decomposition yields 16 terms, each of them having at least one product of the form (2.34) or (2.35), i.e. with opposite terms: $(\mathbf{I})(\mathbf{III})$ and $(\mathbf{II})(\mathbf{IV})$. If we now bound the other factors by one, we get $\sum_n \sqrt{\alpha_n}$ in (2.33), while our assumption only ensures that $\sum_n \alpha_n < \infty$. To get the missing factor $\sqrt{\alpha_n}$ we have to be a bit more careful. First, obviously, terms of the form $(\mathbf{I})^2(\mathbf{III})^2$, $(\mathbf{II})^2(\mathbf{IV})^2$ and $(\mathbf{I})(\mathbf{II})(\mathbf{III})(\mathbf{IV})$ will directly yield the desired α_n. It remains to study terms of the form $(\mathbf{I})^2(\mathbf{II})(\mathbf{III})$, $(\mathbf{I})^2(\mathbf{II})(\mathbf{IV})$, and $(\mathbf{I})^2(\mathbf{II})(\mathbf{IV})$, and the 9 remaining terms beginning with $(\mathbf{II})^2$, $(\mathbf{III})^2$, and $(\mathbf{IV})^2$. Let us treat the first case, the other two terms being similar. Note that $(\mathbf{I})^2 = \sum_{x_1\leq R, y_1\leq 0} \|\mathbf{1}_{X_1}\varphi_n\|^2$, with $X_1 = (x_1, y_1)$. Then going back to (2.20), with obvious notations, we have

$$\begin{aligned}
(\mathbf{I})^2(\mathbf{II})(\mathbf{III}) \ &\leq \sum_{X_1,X_2,X_3} \|\mathbf{1}_{X_1}\varphi_n\|^2\|\mathbf{1}_{X_2}\varphi_n\|\|\mathbf{1}_{X_3}\varphi_n\| && (2.38)\\
&\leq \sum_{X_1,X_2,X_3} (\|\mathbf{1}_{X_1}\varphi_n\|\|\mathbf{1}_{X_2}\varphi_n\|)(\|\mathbf{1}_{X_1}\varphi_n\|\|\mathbf{1}_{X_3}\varphi_n\|)\\
&\leq (c_0\alpha_n)^2 \sum_{X_1,X_2,X_3} \langle X_1\rangle^{4\nu}\langle X_2\rangle^{2\nu}\langle X_3\rangle^{2\nu}\\
&\quad \times \mathrm{e}^{-\frac14(|y_1|^\varsigma-|y_2|^\varsigma-|x_1-x_2|^\varsigma)}\mathrm{e}^{-\frac14(|x_1|^\varsigma-|x_3|^\varsigma-|y_1|^\varsigma-|y_3|^\varsigma)}\\
&\leq C(R)\alpha_n^2. && (2.39)
\end{aligned}$$

□

3. Localization for the Landau operator with a half-plane random potential

We describe some results concerning the localization properties of the Hamiltonians $H(V_1,V_2)$ of interest to the IQHE. First, we sketch the proof of localization for $H(0,V_2)$, with V_2 random, in the large B regime, a result mentioned in Section 2.2 and announced in [CG, Remark 12]. We then sketch the proof of localization for $H(V_1,V_2)$, where V_1 is a left confining potential, and V_2 is a random Anderson-type potential in the large disorder regime and with a covering condition on the single-site potentials. In the large disorder regime, this provides an example of edge currents without edges. Other results for such special models of interest to edge conductance and the IQHE are discussed in [CGH].

3.1. **A large magnetic field regime.**

The aim of this section is to justify Remark 12 in [CG] where localization for $H(0,V)$ away from the Landau levels is claimed. We let $X=(x,y)\in\mathbb{R}^2$, and consider, for

$\lambda > 0$, $B > 0$ given, the Hamiltonian

$$H_\omega = H(0, V_\omega), \text{ with } V_\omega = \lambda \sum_{i \in \mathbb{Z}^+ \times \mathbb{Z}} \omega_i u(X - i). \tag{3.40}$$

The assumptions on the random variables ω_i, $i \in \mathbb{Z}^+ \times \mathbb{Z}$, and on the single site potential u are the one considered in [CH, GK2]. Namely, the ω_i's are i.i.d. random variables with a common law $\mu(\mathrm{d}t) = g(t)\mathrm{d}t$, where g is an even bounded function with support in $[-M, M]$, $M > 0$, with, in addition, the condition $\mu([0,t]) \geq c\min(t, M)^\zeta$, for some $\zeta > 0$[1]. In order to apply the percolation estimate as in [CH] we require that

$$\text{supp } u \in B(0, 1/\sqrt{2}). \tag{3.41}$$

With no loss we assume that $\|u\|_\infty = 1$, so that the spectrum of H_ω satisfies

$$\sigma(H_\omega) \subset \bigcup_{n \geq 1} [B_n - M, B_n + M].$$

We note that thanks to the ergodicity of H_ω with respect to integer translations in the y-direction, the spectrum equals a deterministic set for almost all $\omega = (\omega_i)_{i \in \mathbb{Z}^+ \times \mathbb{Z}}$. For convenience we shall extend the ω_i's to the left half plane by setting $\omega_i = 0$ if $i \in \mathbb{Z}^- \times \mathbb{Z}$.

The only difference between the present model and that of [CH, GK2] is that the random potential in the left half-plane is replaced by a zero potential. This absence of a potential creates a classically forbidden region in the spectral sense for energies between Landau levels. This situation is different from a classically forbidden region created by a wall. The intuition is that looking at a given distance of a Landau level (in the energy axis), the absence of potential should help for localization. One may think of [CH, Wa, GK2]'s result as a weak disorder result. The disorder is kept fixed and localization is obtained for large B. In this spirit putting $\omega_i = 0$ should be even better, for one creates fewer states at a given distance from the Landau level. One might think that the interface at $x = 0$ between the random potential in the right half-plane and the absence of potential in the left one would create some current along the interface. It could be so for energies very close to the Landau level where the above reasoning breaks down.

To get localization, one has to investigate how the Wegner estimate, the multiscale analysis (MSA), and the starting estimates of the MSA are affected by the new geometry of the random potential. In particular, since we broke translation invariance in the x direction, we have to check things for all boxes, regardless of the position with respect to the interface $x = 0$.

The Wegner estimate: It is immediately seen that the proof of the Wegner estimate given in [CH] is still valid with this geometry. Indeed, if a box $\Lambda_L(x, y)$ is such that $x < L/2$, then $\Lambda_L(x, y)$ overlaps the left half-plane (it may even be contained in it). Then, in [CH, (3.8)] the sum is restricted to sites $i = (i_1, i_2)$ where $\omega_i \neq 0$ (i.e. $i_1 > 0$). The rest of the proof is unchanged, and as a result the volume factor one gets at the end is $|\Lambda_L(x, y) \cap (\mathbb{R}^+ \times \mathbb{R})|$ rather than $|\Lambda_L(x, y)|$. In particular one gets zero if $\Lambda_L(x, y) \subset \mathbb{R}^- \times \mathbb{R}$, as expected. So (W) and (NE) of [GK1] hold.

[1] this last hypothesis is not necessary to prove localization at a given fixed distance, independent of B, from the Landau levels

The multiscale analysis: The deterministic part of the MSA (properties (SLI) and (EDI) in [GK1]) is not sensitive to changes of the random variables. Independence of far separated boxes (property (IAD) in [GK1]) is still true. In fact, what happens in the probabilistic estimates that appear in the MSA is that we shall estimate probabilities of bad events related to boxes which have an overlap with the left half-plane as if they where contained in the right half-plane, and thus by a bigger (thus worse) probability. In particular, if a box is totally included in the left half-plane, the probability of having a singular box is zero, and we shall estimate it by a polynomially (or sub-exponentially) small factor in the size of the box.

The starting estimate: We follow the argument given in [CH]. Let us focus on energies $E \in]B_n, B_n + M]$, the other case $E \in [B_n - M, B_n[$ being similar. We thus set $E = B_n + 2a$, $a > 0$. We say that a site $i \in \mathbb{Z}^2$ is *occupied* if $\omega_i \in [-M, a]$, in other words, $\text{dist}(E, B_n + \omega_i) \geq a$ (recall $\|u\|_\infty$=1). Note that by hypothesis on ω_i, for any $a > 0$,

$$\mathbb{P}(\omega_i \in [-M, a]) \geq \frac{1}{2} + ca^\zeta.$$

In particular, the probability is $\mathbb{P}(\omega_i \in [-M, a]) = 1$, if $i \in \mathbb{Z}^- \times \mathbb{Z}$. We are thus above the critical bond percolation threshold $p_c = \frac{1}{2}$ (in dimension 2) for all $i \in \mathbb{Z}^2$. Consequently, bonds percolate, and [CH, Proposition 4.1] follows. The rest of the proof leading to the initial length scale estimate [CH, Proposition 5.1] is the same.

At this stage Theorem 4.1 in [GK2] applies, and one has Anderson localization, (SULE), and strong Hilbert-Schmidt dynamical localization as described in [GK1], as well as (SUDEC) (following the proof of [Ge]; see also [GK3]).

We note that the above arguments are not restricted to the particular half-plane geometry of the random potential we discussed here. Any random potential of the form $V_\omega = \sum_{i \in \mathcal{J}} \omega_i u(X - i)$, where $\mathcal{J} \subset \mathbb{Z}^2$ has an infinite cardinal would yield the same localization result.

3.2. **A large disorder regime.**

We next consider the random Landau Hamiltonian defined in (3.40) with a left constant confining potential $V_0(x, y) = V_0 \mathbf{1}_-$ (see (1.2)) so that $H(V_0, V_\omega) = H_L + V_0 \mathbf{1}_- + \lambda V_\omega \mathbf{1}_+$, for large values of the disorder parameter λ. The random potential in the right half-plane V_ω, as in (3.40), has i.i.d. random variables $\omega_i' s$ with a common *positively* supported distribution, say on $[0, 1]$. We also impose the condition that the single site potential $u \in \mathcal{C}_c^\infty(\mathbb{R})$ satisfies the following covering condition: If $\Lambda \subset \mathbb{R}^+ \times \mathbb{R}$,

$$\sum_{i \in \Lambda} u(X - i) \geq C_0 \mathbf{1}_\Lambda. \tag{3.42}$$

We show that if the disorder is large enough, then at low energy, no edge current will exist along the interface $x = 0$ in the sense that the regularized edge conductance $\sigma_e^{reg}(g, V_0, \lambda V_\omega)$ of $H(V_0, V_\omega)$ will be zero. As consequence of (2.2), however, the regularized edge conductance of $H(0, V_\omega)$ will be quantized to a non zero value, i.e. $\sigma_e^{reg}(g, 0, \lambda V_\omega) = -N$. In other terms the random potential λV_ω is strong enough to create "edge currents without edges" (as in [EJK1]). Such a situation is similar to the model studied by S. Nakamura and J. Bellissard [NB], and revisited in [CG] from the "edge" point of view.

The strategy to prove localization for $H(V_0, V_\omega)$ is the same as the one exposed in Section 3.1, i.e. use a modified multiscale analysis taking into account the new geometry of the problem. Here the potential in the left half-plane is no longer zero but a constant $V_0 > b$, if $I = [0, b]$ is the interval where we would like to prove localization. As in Section 3.1, the modifications of the Wegner estimate, of the starting estimates of the the multiscale analysis (MSA), and of the MSA itself, have to be checked separately. While the comments made in Section 3.1 concerning the MSA are still valid, the new geometry requires new specific arguments for the Wegner estimate and the starting estimate.

The Wegner estimate: Its proof can no longer be borrowed from [CH] as in Section 3.1, and one has to explicitly take into account the effect of the confining potential $V_0 \mathbf{1}_-$. We shall modify the argument given in [CHN] as follows. The only case we have to discuss is the one of a box $\Lambda_L(X)$, with $|x| < \frac{L}{2}$ so that it overlaps both types of potentials. We set

$$\tilde{V}_L = \sum_{i \in \Lambda_L(X) \cap \mathbb{Z}^2} \omega_i u(X - i).$$

Let H_L denote the restriction of $H(V_0, V_\omega)$ to the box $\Lambda_L(X)$ with self-adjoint boundary conditions (e.g. [GKS]). By Chebychev's inequality, the proof of the Wegner estimate is reduced to an upper bound on the expectation of the trace of the spectral projector $E_{H_L}(I)$ for the interval I. Following [CHK], we write

$$\begin{aligned} \mathrm{tr} E_{H_L}(I) &= \mathrm{tr} \mathbf{1}_{\Lambda_L(X)} E_{H_L}(I) && (3.43) \\ &\leq \frac{1}{V_0} \mathrm{tr} V_0 \mathbf{1}_{\Lambda_L(X)} \mathbf{1}_- E_{H_L}(I) + \frac{\lambda}{C_0} \mathrm{tr} \tilde{V}_L E_{H_L}(I) && (3.44) \\ &\leq \frac{1}{V_0} \mathrm{tr} \mathbf{1}_- H_L E_{H_L}(I) + \frac{\lambda}{C_0} \mathrm{tr} \tilde{V}_L E_{H_L}(I) && (3.45) \\ &\leq \frac{b}{V_0} \mathrm{tr} E_{H_L}(I) + \frac{\lambda}{C_0} \mathrm{tr} \tilde{V}_L E_{H_L}(I), && (3.46) \end{aligned}$$

so that, with $V_0 > b$ by assumption,

$$\mathrm{tr} E_{H_L}(I) \leq \frac{\lambda}{C_0} \left(1 - \frac{b}{V_0}\right)^{-1} \mathrm{tr} \tilde{V}_L E_{H_L}(I). \quad (3.47)$$

At this point, the proof follows the usual strategy, as in [CHN, CHKN, CHK].

The starting estimate: The initial estimate follows from the analysis, at large disorder, given in [GK2, Section 3]. Since in the left half-plane, the potential is already very high ($V_0 > b$), it is enough to estimate the probability that all the random variables ω_i in the right part of the box is higher than say $b/2$. Doing this creates a gap in the spectrum of the finite volume operator H_L. This spectral gap, occurring with good probability, can be used to obtain the exponential decay of the (finite volume) resolvent thanks to a Combes-Thomas argument.

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